

The Willmore functional on complete minimal surfaces in \mathbb{H}^3 : boundary regularity and bubbling.

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Abstract

We study various aspects related to boundary regularity of complete properly embedded minimal surfaces in \mathbb{H}^3 , particularly those related to assumptions on boundedness or smallness of Willmore energy. We prove, in particular, that small energy gives control on \mathcal{C}^1 boundary regularity. We examine the possible lack of convergence in the \mathcal{C}^1 norm for sequences of finite energy minimal surfaces; we find that the mechanism responsible for this is a bubbling phenomenon of energy escaping to infinity.

1 Introduction

In our previous paper [1] we studied the renormalized area, $\text{RenA}(Y)$, as a functional on the space of all properly embedded minimal surfaces Y in \mathbb{H}^3 with a sufficiently smooth boundary curve at infinity. Area or volume renormalization of a minimal submanifold of arbitrary dimension or codimension in hyperbolic space was introduced by Graham and Witten [9]; the renormalization is accomplished by an Hadamard regularization of the asymptotic expansion of areas (or volumes) of a family of compact truncations of the submanifold. The renormalized area of a surface in \mathbb{H}^3 turns out to be a classical quantity. The first result in [1] is that

$$\text{RenA}(Y) = -2\pi\chi(Y) - \frac{1}{2} \int_Y |\mathring{A}|^2 d\mu, \quad (1.1)$$

where $\chi(Y)$ is the Euler characteristic and \mathring{A} the trace-free second fundamental form of Y . Since Y is minimal, $\int_Y |\mathring{A}|^2 d\mu$ is the total curvature of the surface Y ; furthermore given the special asymptotic behaviour of the minimal surfaces Y with \mathcal{C}^2 boundary near $\partial_\infty \mathbb{H}^3$, it follows from Gauss-Bonnet and the conformal invariance of $|\mathring{A}|^2 d\mu$ that if we let $\int_Y |\bar{A}|^2 d\bar{\mu}$ be the total curvature of Y with respect to the Euclidean metric in the upper half-space model \mathbb{R}_+^3 , then

$$\int_Y |\bar{A}|^2 d\bar{\mu} = \int_Y |\mathring{A}|^2 d\mu + 4\pi\chi(Y) = \int_Y |\mathring{A}|^2 d\mu + 4\pi\chi(Y). \quad (1.2)$$

It is thus natural to think of this functional as a Willmore-type energy for the space of complete minimal surfaces in \mathbb{H}^3 . From now on we refer to $\mathcal{E}(Y) := \int_Y |\mathring{A}|^2 d\mu$ as the energy of Y .

The aim of this paper is to study convergence properties of sequences of minimal surfaces with a given genus and number of ends, with energy bounded above. This is linked to understanding the implication of finite total energy on the regularity of

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the asymptotic boundary. Thus, given such a sequence Y_j , with $\mathcal{E}(Y_j) \leq C < \infty$, we study the possibilities for (sub)convergence of this sequence in \mathcal{C}^1 , which turns out to be the strongest norm we can control, as well as the mechanism responsible for failure of \mathcal{C}^1 convergence via loss of energy in the limit. Since the theory of convergence of minimal surfaces on any compact set is well understood, our focus is almost entirely on the uniform or limiting behavior of these surfaces near and at the boundary $\partial_\infty \mathbb{H}^3$.

Before stating our results, we put this into a broader context. The study of failure of compactness for variational problems goes back at least to [23] and has now been explored in a wide variety of settings; we refer to [21] for a good overview of results and methods. Particularly relevant to our problem are the many deep advances in understanding the analytic aspects of the Willmore functional; we refer in particular to the fundamental paper of L. Simon [24], the more recent work by Kuwert and Schätzle [13] and the powerful new approach developed by Rivière [20], see also [18]. However, none of this work (on Willmore or otherwise), to our knowledge, deals with this loss of compactness due to bubbling at the boundary. Often this possibility is excluded by imposing apriori bounds on the regularity of the boundary. Our particular geometric problem presents a natural situation where this phenomenon occurs. In another direction, the issue of regularity up to the boundary for prescribed mean curvature submanifolds with *unprescribed* boundary has been studied in the Euclidean setting in [12].

The second context in which to view our work is slightly more tenuous. To explain it we first recall the computation from [1] which gives the first variation of \mathcal{E} . If $\gamma = \partial_\infty Y$ is the boundary curve at infinity, then there is function u_3 associated to the minimal surface Y such that

$$D\mathcal{E}|_Y(\psi) = 6 \int_\gamma u_3 \psi_0 ds. \quad (1.3)$$

Here ψ is a Jacobi field along Y , i.e. an infinitesimal variation of Y amongst minimal surfaces and ψ_0 its boundary value at γ , and s is the arclength parameter along γ . We made a case in [1] that the pair (γ, u_3) should be regarded as the Cauchy data of Y . It follows from the basic regularity theory for such surfaces, due to Tonegawa [25], that if the ‘Dirichlet data’ γ is \mathcal{C}^∞ , then \overline{Y} is \mathcal{C}^∞ up to the boundary. Based on classical elliptic theory, one might also expect that control on the Neumann data, u_3 , should also yield regularity of Y up to the boundary. In particular, if Y_j is a Palais-Smale sequence for \mathcal{E} , then the functions $u_3^{(j)}$ converge to zero in some weak sense, and the question then becomes whether quantitative measures of smallness on these functions yield greater control on the boundary curves γ_j . We do not emphasize this point of view, however, since it is difficult to make precise.

Results: We first prove an ϵ -regularity result: if the energy of a surface in a (Euclidean) half-ball (in the upper half-space model) around some point $P \in \partial_\infty Y$ is small, then the energy controls the \mathcal{C}^1 norm of the surface uniformly up to the boundary. This has the following analytic content: regarding Y as a horizontal graph over a vertical half-plane, then finiteness of the energy is slightly weaker than bounding the graph function in $W^{2,2}$. Hence \mathcal{C}^1 regularity shows that this graph function exhibits *better* behavior near the boundary than would follow from the Sobolev embedding theorem or more direct PDE arguments using the minimal surface equation. This \mathcal{C}^1 regularity is nearly optimal. Indeed, the energy \mathcal{E} is dilation-invariant, but if we take a blow-down limit of a given surface, then the $\mathcal{C}^{1,\alpha}$ norm of the boundary curve diverges, so we could not expect that norm to be controlled by the energy. It is not clear at this stage how to characterize the optimal regularity associated with finiteness of energy, but this is certainly an interesting question. It is also not obvious what lies at the heart of this unexpected extra boundary regularity enjoyed by these minimal surfaces, but this seems to be a non-linear phenomenon which relies crucially on the geometric nature of the problem. Our methods (especially in §5)

seem to indicate that it is essential that we are dealing with surfaces with boundary only at infinity.

One application of this first result is that if Y_j is a sequence of minimal surfaces with bounded energy and well-separated boundary components, then some subsequence converges to a minimal surface Y_* , the boundary at infinity of which is a priori Lipschitz except at a finite number of bad points. We then show that except possibly at these exceptional points, the limit curve is \mathcal{C}^1 . We refer to this extra regularity (compared to Sobolev embedding) as the gain of regularity. We note that the convergence of $\gamma_j = \partial_\infty Y_j$ to $\gamma_* = \partial_\infty Y_*$ need not be \mathcal{C}^1 ; in fact we construct examples of this at the end of this paper: Using fairly simple gluing arguments, we obtain a sequence Y_j with energy $\mathcal{E}(Y_j) \leq C$ which converges to a totally geodesic hemisphere, but where the convergence is not \mathcal{C}^1 at a finite number of boundary points. At each of those points, one sees a sequence of increasingly strong blow-downs of a fixed minimal surface, which carries a fixed positive amount of energy, shrink to a point; we regard this as a type of bubbling. However, unlike the various ‘interior’ bubbling we referred to earlier, in this setting there is no quantization of energy; arbitrarily small amounts of energy can disappear in these limits. We emphasize that the points at infinity to which these dilated surfaces converge, and in general the points where the convergence $\gamma_j \rightarrow \gamma_*$ is not \mathcal{C}^1 even though all curves are \mathcal{C}^1 , are *not* the bad points which appear in our ϵ -regularity theorem. Indeed it is possible that the convergence of boundary curves might be only in $\mathcal{C}^{0,\alpha}$ everywhere, even when γ_* is smooth.

Our final result is that the phenomenon exhibited by these examples above is in fact the only mechanism through which the convergence $Y_j \rightarrow Y_*$ can fail to be \mathcal{C}^1 near the boundary, at least in regions of small energy. In such regions we show that if $P_j \in \gamma_j$, $P_j \rightarrow P_* \in \gamma_*$, but the tangent lines $T_{P_j} \gamma_j$ fail to converge to $T_{P_*} \gamma_*$, then there exist a sequence of hyperbolic isometries φ_j which dilate away from P_j and are such that $\varphi_j(Y_j) \rightarrow \tilde{Y}_*$, where $\mathcal{E}(\tilde{Y}_*) > 0$. The investigation of gain of regularity and bubbling in regions of large energy presents various technical difficulties (some of which are already apparent in [14]) that are beyond the scope of this paper. We intend to return to this in a subsequent paper.

We now provide a brief outline of some of the key ideas and arguments in this paper. The preamble to each section contains more extensive discussion of the main idea in that section.

Outline: The argument commences in §2, where we prove two “soft” results about boundary regularity for minimal surfaces with finite energy. Together, these show that any such surface must meet $\partial_\infty \mathbb{H}^3$ orthogonally and have a good local graphical representation over a vertical plane provided the boundary curve has a corresponding graphical representation over a line. This relies only on interior regularity results for minimal surfaces and simple Morse-theoretic arguments.

ϵ -regularity: In our first “hard” result, we prove that for (local) minimal surfaces with small enough energy, one obtains \mathcal{C}^1 control on the boundary curve. Indeed, we argue that if this were to fail, then one could construct a sequence of minimal surfaces, the energies of which vanish in the limit, but such that there is a jump in the tangent lines in the limit. To reach a contradiction, we wish to relate the slope of the tangent line at the boundary to information on a parallel curve in the interior of the surface and then use the known \mathcal{C}^∞ convergence in the interior.

The relationship between derivative information in the interior and at the boundary, i.e. the difference between the ‘horizontal’ derivatives at height 0 and 1, say, is given by integrating the mixed second derivative of the graph function along a vertical line and showing that this is controlled by the energy. To do this we must use a choice of ‘gauge’, which is a special isothermal coordinate system for which we have explicit pointwise control of the conformal factor. Using some deep results in harmonic analysis, such coordinate systems have been obtained for related problems, e.g. for embedded spheres by De Lellis and Müller [8], following an earlier and very

influential paper by Müller and Sverak [19], see also Hélein [11]; We must modify those arguments to our setting, which requires a ‘preparation’ of our surface in a couple of ways. We first locally graft our surfaces into a round sphere so that the resulting non-round sphere has two reflection symmetries. We then apply an appropriate Möbius transformation to normalize the positioning of this surface so as to be in a position to apply the results in [8], [19]. Throughout this whole procedure we must be careful that none of these alterations change the fact that there is a jump in first derivatives at the origin. Finally, in these isothermal coordinates, we use the fact that the mixed component \dot{A}_{12} of the second fundamental form is a harmonic function; this is a special feature of minimal surfaces in constant curvature three-manifolds. This harmonicity leads, via a monotonicity formula, to the fact that this component decays somewhat more quickly than was known before, which leads eventually to partial control of the line integral mentioned above. In this argument there is a second line integral which it is necessary to control in terms of the energy of Y_j in a half-ball. This second line integral plays a crucial role in the later analysis of bubbling.

These arguments occupy §3-5. In §6 we use the techniques developed up to that point to derive the regularity gain for the limit surface Y_* in regions of small energy.

Bubbling: Section 7 contains the argument that if the convergence $Y_j \rightarrow Y_*$ is not \mathcal{C}^1 at some sequence of boundary points $P_j \rightarrow P_* \in \partial_\infty Y_*$, then we can perform a sequence of blowups near those boundary points to produce a sequence of minimal surfaces \tilde{Y}_j which converge to a limit surface \tilde{Y}_* of non-zero energy; prior to the blow-up the surfaces \tilde{Y}_j are disappearing in the limit towards P_* . In other words, the \mathcal{C}^1 loss of compactness is due to portions of Y_j with fixed (but arbitrary) nonzero energy disappearing at infinity. Unlike similar arguments for bubbling in the interior, since our surfaces have infinite area, it is not initially clear that we can find points $Y_j \ni Q_j \rightarrow \partial_\infty \mathbb{H}^3$ on which $|\dot{A}|_g$ is bounded below; the rescalings we wish to perform should be centered at these points. Their existence is proved indirectly, by arguing that it is impossible for all possible blowups near the points $P_j \in Y_j$ to converge to surfaces of zero energy. This argument makes essential use of the second line integral mentioned above. The key point is to show that this line integral can be controlled by the energy of Y_j in a conical region emanating from (rather than a half-ball containing) P_j .

Further questions: There are several questions and problems which are closely related to the themes in this paper and which seem particularly interesting. We hope to return to some of these soon.

Despite the fact that the problems which led us to the current investigations involve minimal surfaces in \mathbb{H}^3 , it is probably the case that the questions here should all be investigated in the slightly broader category of properly embedded Willmore surfaces in \mathbb{H}^3 with finite energy. To be clear, consider the total curvature functional

$$\tilde{\mathcal{E}}(Y) = \int_Y |A|^2 d\mu,$$

and the class of surfaces Y for which $\tilde{\mathcal{E}}(Y) < \infty$ and which are critical for this functional. We call these complete, finite energy Willmore surfaces in \mathbb{H}^3 . Note that minimal surfaces Y with $\mathcal{E}(Y) < \infty$ automatically fall into this class. The analysis of boundary regularity and bubbling is interesting in this broader context too. Although the methods in this paper use the minimality of the surfaces in an essential way, we conjecture that the picture developed here concerning extra regularity and bubbling should extend to finite energy complete Willmore surfaces in \mathbb{H}^3 as well.

Another question, which certainly also motivated this work, but which is not the focus of any of the results here, concerns the analysis of sequences of minimal surfaces Y_j which are Palais-Smale for the functional \mathcal{E} . Recall that this means that $\mathcal{E}(Y_j)$ tends to a critical value and $D\mathcal{E}|_{Y_j}$ converges to 0. The goal would be to find

critical points for \mathcal{E} , or indeed, RenA. Our results show that critical sequences may converge to surfaces with strictly lower genus, and this convergence will often be in only a weak norm at the boundary, but it may still be possible to produce \mathcal{E} -critical surfaces this way.

Finally, one other set of problems we wish to mention involve an analogous though more complicated problem of studying sequences of Poincaré-Einstein metrics in four dimensions. Recall that (M, g) is said to be Poincaré-Einstein if M is a compact manifold with boundary, and g is conformally compact (hence is complete on the interior of M) and Einstein, see [16, 3] for more details and further references. These objects can be studied in any dimension, but it is known that dimension 4 is critical in the same way that dimension 2 is critical for minimal surfaces. This is reflected in some formulæ due to Anderson [2], the first of which gives an explicit local integral expression for the renormalized volume of a four-dimensional Poincaré-Einstein space as the sum of an Euler characteristic and the squared L^2 norm of the Weyl curvature, and another, which describes the differential of renormalized volume with respect to Poincaré-Einstein deformations. These are quite similar to (and indeed motivate) the analogous formulæ here. It is therefore reasonable to ask whether results like the ones here can be proved in that setting. More specifically, suppose that (M^4, g_j) is a sequence of Poincaré-Einstein metrics for which one only has control of $\int |W_j|^2 dV_{g_j}$, where W_j is the Weyl tensor of g_j . To what extent does this control the regularity of the sequence of conformal infinities of g_j , which are conformal classes on ∂M ? This is related to the questions studied by Anderson [3] and Chang-Qing-Yang [4].

1.1 Results

Almost all of what we do is local, so we always work in the upper half-space model \mathbb{R}_+^3 of \mathbb{H}^3 , with vertical (height) coordinate x , and with linear coordinates (y, z) on $\mathbb{R}^2 = \{x = 0\}$. We introduce some nomenclature and notation.

First, since dilation in \mathbb{R}_+^3 is a hyperbolic isometry, we fix a normalization and scale. We say that Y is normalized if it is connected, has boundary curve γ with length $|\gamma| = 100\pi$ and if the center of mass of γ in \mathbb{R}^2 is the origin. We also assume that γ is at least \mathcal{C}^2 unless explicitly stated otherwise. The class of all such normalized surfaces with k ends and genus g will be denoted $\mathcal{M}_{k,g}$. We let $\mathcal{M} = \bigcup_{k,g} \mathcal{M}_{k,g}$. For each $Y \in \mathcal{M}$ we let \bar{Y} stand for the closure of Y in \mathbb{R}_+^3 .

Definition 1.1. Fix any $Y \in \mathcal{M}$; if P is a point in γ and $R > 0$, then $B(P, R)$ is the open Euclidean half-ball centered at P of radius R . Denote by $Y'_{B(P,R)}$ the path component of $\bar{Y} \cap B(P, R)$ which contains P . Let $\gamma'_{B(P,R)}$ be the boundary at infinity of $Y'_{B(P,R)}$. Define the localized energy of $Y'_{B(P,R)}$ by

$$\mathcal{E}^{B(P,R)}(Y) := \int_{Y'_{B(P,R)}} |\mathring{A}|^2 d\mu. \quad (1.4)$$

We first record a trivial consequence of the Gauss-Bonnet formula, which only relies on the boundary regularity results from [10, 15, 25] for minimal surfaces with \mathcal{C}^2 boundary.

Lemma 1.1. Suppose that Y is a complete properly embedded surface which is \mathcal{C}^2 in $\{x \geq 0\}$ and suppose that the closure \bar{Y} intersects $\partial_\infty \mathbb{H}^3$ orthogonally, then the Gauss-Bonnet formula gives

$$\int_Y |\bar{A}|_g^2 d\bar{\mu} = \mathcal{E}(Y) + 4\pi\chi[Y].$$

In particular, the functional \mathcal{E} is essentially the same as the Euclidean total curvature for such surfaces. It also follows from this that the Euclidean total curvature is invariant under hyperbolic isometries. For future reference we let $\mathcal{E}'(Y) = \int_Y |\bar{A}|_g^2 d\bar{\mu}$.

Next, fix any closed embedded \mathcal{C}^1 curve $\gamma \subset \mathbb{R}^2$. Define the ζ -Lipschitz radius of γ as follows. If $P \in \gamma$ and $\ell_\gamma(P)$ is the tangent line to γ at P , then let $\gamma_P \subset \gamma$ be the largest open connected arc containing P which is a graph over $\ell_\gamma(P)$. Thus, assuming $P = 0$ and $\ell_P = \{(0, y, 0)\}$, we can write $\gamma_P = \{(y, f(y)) : a < y < b\}$ for some maximal $a < 0 < b$. We then define $\text{LipRad}_\gamma^\zeta(P)$ to be the largest number M such that $(-M, M) \subset (a, b)$ and $\frac{|f(y) - f(y')|}{|y - y'|} < \zeta$ for every $y, y' \in (-M, M)$. Finally, we set

$$\text{LipRad}^\zeta(\gamma) = \inf_{P \in \gamma} \text{LipRad}_\gamma^\zeta(P). \quad (1.5)$$

Since γ is compact and LipRad^ζ is easily seen to be lower-semicontinuous, $\text{LipRad}_\gamma^\zeta > 0$ and the infimum is attained at some point.

We can now state our first main ϵ -regularity result. All our ϵ -regularity results apply to the space of surfaces in $\mathcal{M}_{k,g}$ with energy bounded above by some number $M < \infty$.

Theorem 1.1. *There is a $\zeta_0, 0 < \zeta_0 < 1/20$ with the property that if $\zeta \in (0, \zeta_0)$, then there exists an $\epsilon(\zeta) > 0$ such that if $Y \in \mathcal{M}_{k,g}$ and $\mathcal{E}^{B(P,R)}(Y) < \epsilon(\zeta)$ for some $P \in \gamma = \partial_\infty Y$ and $R \leq 1$, then*

$$\text{LipRad}_\gamma^\zeta(Q) \geq \zeta \cdot \frac{R - |PQ|}{10}$$

for all $Q \in \gamma'_{B(P,R)}$.

Using this result and Lemma 2.4 below, we can deduce the

Corollary 1.1. *In the setting of Theorem 1.1, there exists $\epsilon'(\zeta) \leq \epsilon(\zeta)$ such that if $\mathcal{E}_Y^{B(P,R)} < \epsilon'(\zeta)$, then the surface $Y'_{B(P,R/2)}$ is a horizontal graph $z = u(x, y)$ over the half-disc $D(P, R/2)$ in the vertical half-plane $\mathbb{R}_+ \times \ell_P$, and $|\nabla u| \leq 2\zeta$ in $D(P, R/2)$.*

The Lipschitz radius is a reasonable measure of regularity on the space of normalized embedded curves γ . Note that if γ_j is a sequence of such curves with $\text{LipRad}(\gamma_j) \geq C > 0$, then there are uniform Lipschitz parametrizations around each point of every γ_j , hence in particular some subsequence of the γ_j converge in $\mathcal{C}^{0,\alpha}$ for any $\alpha < 1$ to a limit curve γ which is itself Lipschitz.

We state a slightly modified version of this result, for future use. Define the space \mathcal{M}' of minimal surfaces $Y \subset \mathbb{H}^3$ whose boundary curves $\gamma = \partial Y$ are \mathcal{C}^1 . For such surfaces, $Y'_{B(P,R)}$ and $\mathcal{E}^{B(P,R)}(Y)$ still make sense. The modification deals with surfaces $Y \in \mathcal{M}'$ for which $\gamma'_{B(P,R)}$ intersects $\partial B(P, R)$.

Theorem 1.2. *For some $\zeta_0 > 0$ and every $\zeta \in (0, \zeta_0)$ there exists an $\epsilon(\zeta) > 0$ such that if $Y \in \mathcal{M}'$, $\mathcal{E}^{B(P,R)} \leq \epsilon(\zeta)$ and $\gamma'_{B(P,R)}$ intersects $\partial B(P, R)$ for some $P \in \gamma = \partial_\infty Y$ and if $\gamma'_{B(P,R)}$ is \mathcal{C}^1 up to its endpoints, then*

$$\text{LipRad}_\gamma^\zeta(Q) \geq \zeta \cdot \frac{R - |PQ|}{10}$$

for all $Q \in \gamma'_{B(P,R)}$.

Theorem 1.1 leads to the following picture on the possible limits of minimal surfaces with energy bounded above:

Theorem 1.3. *Let $Y_j \in \mathcal{M}_{k,g}$ be a sequence of surfaces with genus g and k ends. Suppose that $\mathcal{E}(Y_j) \leq M < \infty$ and that there is a minimum positive separation, independent of j , between the various components of the boundary curves γ_j of each Y_j . Then, given any $0 < \zeta \leq \zeta_0$, there is a subsequence, which we again denote Y_j , converging with finite multiplicity to a complete properly embedded (possibly disconnected) minimal surface Y_* with boundary curve γ_* . The convergence is smooth away from $\{x = 0\}$, except at a finite number of interior points.*

Furthermore, there exists a finite number of points $P_1, \dots, P_N \in \gamma_*$, $N = N(\zeta)$, and corresponding sequences points $P_i^{(j)} \in \gamma_j$, $i = 1, \dots, N$, with $P_i^{(j)} \rightarrow P_i$ for all i , such that the convergence of γ_j to γ_* is $C^{0,\alpha}$ for every $\alpha < 1$ away from the points $P_i^{(j)}$. Finally, if $P \in \gamma \setminus \{P_1, \dots, P_N\}$, then there is a line ℓ_P such that Y_* is the graph of a Lipschitz function with Lipschitz constant 2ζ over some disc in the half-plane over ℓ_P .

This Theorem can be deduced in a straightforward way from Theorem 1.1 and Corollary 1.1, together with standard arguments regarding the interior convergence. Indeed, pick out the largest half-ball $B(Q, \delta(Q))$ around each point $Q \in \gamma_j$ such that $\mathcal{E}(Y'_{j,B(Q,\delta(Q))}) \leq \epsilon'(\zeta)$. The uniform upper bound on $\mathcal{E}(Y_j)$ implies that in terms of the arc-length parameter t of (each component of) γ_j , for all but finitely many values of t (corresponding to the points $Q_i^{(j)}$ in the statement of the theorem), $\liminf_j \delta(\gamma_j(t)) > 0$. Thus using a diagonalization argument we prove the claim regarding convergence near the boundary. The smooth convergence in the interior (except at a finite number of points) follows from [6] (see also [22]), except for the claim on finite multiplicity. To see this, observe that the finiteness of the total energy implies that if some component of $Y_{*,a}$ of Y_* has infinite multiplicity, then Y_* would have to be totally geodesic, hence the length of $\partial Y_{*,a}$ (counted with multiplicity) would be infinite, which is impossible.

We next prove a gain of regularity result, which in fact shows that the limit curve γ_* is not just Lipschitz but is actually C^1 away from a finite set of points. In other words each component of $\gamma_* = \partial Y_*$ is a union of C^1 arcs which meet at this finite set $\{P_1, \dots, P_N\}$.

Theorem 1.4. *In the setting of theorem 1.3, the boundary $\gamma_* = \partial_\infty Y_*$ is piecewise C^1 , and (possibly) fails to be C^1 at most at the set $\{P_1, \dots, P_N\} \subset \gamma_*$.*

Remark 1.1. *By a simple modification of the proof of Lemma 2.4 below, the surface \overline{Y}_* is then C^1 up to $\gamma_* \setminus \{P_1, \dots, P_N\}$.*

We also describe bubbling in this setting by showing that away from the bad points, at all points where the convergence $\gamma_j \rightarrow \gamma_*$ fails to be C^1 , the loss of compactness is due to some portion of the minimal surfaces with non-zero energy disappearing towards infinity.

Theorem 1.5. *Let Y_j be a sequence of minimal surfaces in $\mathcal{M}_{k,g}$ with $\mathcal{E}(Y_j) \leq M < \infty$ and such that $Y_j \rightarrow Y_*$ where Y_* is C^1 up to $\gamma_* \setminus \{P_1, \dots, P_N\}$. After rotation and translation, Y_j is a horizontal graph $z = u_j(x, y)$ over the half-disc $\{x^2 + y^2 \leq \delta^2\}$, with $|\nabla u_j| \leq 2\zeta$ and $u_j \rightarrow u_*$ in C^∞ away from $\{x = 0\}$ and in $C^{0,\alpha}$ up to $\{x = 0\}$. Finally, suppose that for some $y_0 \in (-\delta, \delta)$, $\lim_{j \rightarrow \infty} \partial_y u_j(y_0, 0) \neq \partial_y u_*(y_0, 0)$. Setting $P_j = (0, y_0, u_j(y_0, 0))$, then there exists a sequence of interior points $Q_j \in Y'_{j,B(0,\delta)}$ with $Q_j \rightarrow P_j$ such that if ψ_j is a hyperbolic isometry which maps Q_j to $(1, 0, 0)$, then $\psi_j(Y_j) \rightarrow Y'_*$ for some complete minimal surface Y'_* with $\mathcal{E}(Y'_*) > 0$.*

Finally, in the last section, we construct examples where this loss of compactness via bubbling to infinity does occur: We construct sequences of minimal surfaces $Y_j \in \mathcal{M}$ with $\mathcal{E}(Y_j) \leq M < \infty$ and with \overline{Y}_j converging smoothly away from a finite number of points on the boundary. At these points, the convergence fails to be C^1 , despite the fact that the curves γ_j and γ_* are all C^1 .

Notation and terminology: We now record some notation and terminology used throughout the paper. Let Y be a minimal surface in \mathbb{H}^3 . In many arguments we use the interplay between the metrics g and \bar{g} on Y induced from the ambient hyperbolic and Euclidean metrics, respectively. We denote by A and \bar{A} , and \mathring{A} and $\bar{\mathring{A}}$ the corresponding second fundamental form and trace-free second fundamental forms of Y and by $d\mu, d\bar{\mu}$ the area elements.

Recall also that if $g = e^{2\phi}\bar{g}$ are two conformally related metrics, then the mean curvatures of any submanifold Y are related by

$$H = e^{-\phi} (\bar{H} + \partial_{\bar{\nu}}\phi), \quad (1.6)$$

where $\bar{\nu}$ is the (Euclidean) unit normal of Y . In particular, if $H = 0$, then

$$\bar{H} = -\partial_{\bar{\nu}}\phi. \quad (1.7)$$

We also have

$$|A|_g^2 d\mu = |\mathring{A}|_{\bar{g}}^2 d\bar{\mu}. \quad (1.8)$$

Finally, if Y is minimal with respect to g , so that $A = \mathring{A}$, then

$$|A|_g^2 d\mu = |\mathring{A}|_g^2 d\mu = |\mathring{A}|_{\bar{g}}^2 d\bar{\mu} \quad (1.9)$$

For brevity, we often omit the subscripts g, \bar{g} below, and thus write $|\mathring{A}|^2$ instead of $|\mathring{A}|_g^2$, etc.

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2 Some geometric lemmas

We begin by proving a few geometric results which will be used throughout this paper. These pertain primarily to *fixed* complete minimal surfaces with finite energy and with boundary at infinity consisting of a finite number of disjoint embedded Lipschitz curves. We prove first that any such minimal surface is asymptotically vertical at $\{x = 0\}$, which is the standard behaviour if the surface is \mathcal{C}^1 up to the boundary, [15]. We then show that if some segment of the boundary curve is graphical with a bounded Lipschitz constant, then a portion of the minimal surface directly above this segment is also graphical, with bounded gradient. The proofs are essentially geometric, involving blowup arguments, though we rely on one analytic fact which is the Choi-Schoen ϵ -regularity theorem for minimal discs. The crucial hypothesis used at several places is the finiteness of the total energy.

Lemma 2.1. *Let Y be a complete minimal surface in \mathcal{M} . Assume $\partial_\infty Y$ consists of a finite collection of embedded Lipschitz curves. Let P_j be a sequence of points in Y converging to a point on $\partial_\infty Y$. If $\bar{\nu}$ denotes the Euclidean unit normal to Y , then $\langle \partial_x, \bar{\nu} \rangle_{\bar{g}}(P_j) \rightarrow 0$ as $j \rightarrow \infty$.*

Proof. Since Y has finite energy, then for any $M > 0$, $\int_{Y \cap \{x \leq Mx(P_j)\}} |\mathring{A}|^2 d\mu \rightarrow 0$. Now, suppose that the assertion is false so that, passing to a subsequence if necessary, $\langle \partial_x, \bar{\nu} \rangle_{\bar{g}}(P_j) \rightarrow \beta \neq 0$. Let $B_1(P_j)$ be the ball of radius 1 around P_j with respect to the metric g . Passing to a further subsequence, we may assume that $B_1(P_j) \cap B_1(P_k) = \emptyset$ for $j \neq k$. Then $\int_{B_1(P_j)} |\mathring{A}|^2 d\mu \rightarrow 0$, since otherwise $\mathcal{E}(Y)$ would be infinite.

Now translate Y horizontally and dilate by the factor $1/x(P_j)$ so that P_j is mapped to $(1, 0, 0)$ and denote by Y_j the resulting sequence of surfaces. Since each Y_j passes through the fixed point $(1, 0, 0)$ and $\int_{Y_j \cap \{x \leq M\}} |\mathring{A}|^2 d\mu \rightarrow 0$ for any $M > 0$, we can invoke the a priori pointwise bounds for $|\nabla^p \mathring{A}_j|$ on a ball of any fixed radius around this fixed point proved by Choi-Schoen [6]. These show that yet a further subsequence of the Y_j converge in the \mathcal{C}^∞ topology on compact sets to a complete minimal surface Y_* .

Since $\mathcal{E}^{B_R(P_j)}(Y_j) \rightarrow 0$ for any $R > 0$, we see that Y_* is totally geodesic, and hence is either a vertical plane or a hemisphere; its slope at $(1, 0, 0)$ equals $\beta \neq 0$, so we must be in the latter case. This shows that there is a fixed constant $R = R(\beta) > 0$ such that if j is large, then the ball $B_R(P_j)$ in Y contains a point Q_j where $T_{Q_j}Y$ is horizontal, i.e. parallel to $\{x = 0\}$.

We can assume (passing again to a further subsequence) that $x(Q_j)$ is strictly monotone decreasing, so a standard minimax argument shows that we may choose a sequence of points $Q'_j \in Y_j$ which are critical points of index one for the function x . In other words, writing Y_j as a graph $x = v(y, z)$ near Q'_j , then v has a saddle at Q'_j . We can therefore translate horizontally and dilate by the factor $1/x(Q'_j)$ to obtain a sequence Y'_j of minimal surfaces which converge locally in \mathcal{C}^∞ to a complete minimal surface Y'_* passing through the point $(1, 0, 0)$. By construction, for any $M > 0$,

$$\int_{Y'_j \cap \{x \leq M\}} |\tilde{A}|^2 d\mu \rightarrow 0. \quad (2.1)$$

Using the Choi-Schoen estimates again, we see that the convergence of Y'_j to Y'_* is \mathcal{C}^∞ near the point $(1, 0, 0)$, hence Y'_* has a horizontal tangent plane at this point. Furthermore, the two principal curvatures at Q'_j relative to the ambient Euclidean metric are $\kappa_1 \geq 0$ and $\kappa_2 \leq 0$, and these inequalities must persist in the limit. This means that Y'_* cannot be a hemisphere. However, (2.1) implies that $\mathcal{E}(Y'_*) = 0$, which yields a contradiction. \square

An almost identical argument proves the

Lemma 2.2. *Let Y be a fixed minimal surface in \mathbb{H}^3 with $\mathcal{E}(Y) < \infty$. Let P_j be a sequence of points in $\partial_\infty Y$ and choose $\delta_j \searrow 0$. Denote by $B_{\delta_j}^+(P_j)$ the Euclidean half-ball centered at P_j and with radius δ_j . Assume that the sequence of dilated translates $\delta_j^{-1}(Y \cap B_1^+(P_j) - P_j)$ converges to a minimal surface \tilde{Y} . Then \tilde{Y} must be a vertical half-plane.*

We next turn to proving local graphicality of any minimal surface of finite energy near points where the boundary curve is Lipschitz.

Lemma 2.3. *Let Y be a complete properly embedded minimal surface in \mathbb{H}^3 with finite energy and such that $\partial_\infty Y = \gamma$ is a finite union of closed embedded rectifiable loops. Define \mathcal{S}_B to be the set of all points $P \in \gamma$ for which there exists a connected subarc $\gamma_P \subset \gamma$ which is a graph over a straight line $\ell_P \subset \mathbb{R}^2$ containing P which, if we rotate and translate so that ℓ_P is the y -axis, has graph function $z = f(y)$, $|y| \leq \delta(P)$, satisfying $\text{Lip}(f) \leq B$.*

Then there exists an $h > 0$, independent of P , such that the portion $Y'_{B(P, h\delta(P))}$ of the surface Y is graphical over the half-disc $\{\sqrt{x^2 + y^2} \leq h\delta(P), z = 0\}$ with graph function $z = u(x, y)$, where u satisfies $|\nabla u| \leq 2B$.

Proof. If this were false, then there would exist a sequence $P_j \in \gamma$, lines ℓ_j and graph functions $f_j : [-\delta_j, \delta_j] \rightarrow \mathbb{R}$ for γ with Lipschitz constant B , and sequences of numbers $h_j \rightarrow 0$ and points $Q_j \in Y'_{B(P_j, h_j\delta_j)}$ with coordinates (x_j, y_j, z_j) (using coordinates (x, y, z) where P_j is the origin and ℓ_{P_j} is the y -axis), such that angle between the unit (Euclidean) normal $\bar{\nu}(Q_j)$ to Y at Q_j and ∂_z is greater than $\pi/2 - \arctan(2B)$.

Since Y has finite energy, we have that $\mathcal{E}^{B(P_j, h_j\delta_j)}(Y) \rightarrow 0$, so a contradiction can be drawn by the usual blow-up argument. Translate so that $y_j = 0$, then dilate by the factor $\frac{1}{x_j}$ to obtain a sequence of surfaces \tilde{Y}_j . By construction, $\partial_\infty \tilde{Y}_j$ is graphical over the y -axis at least over the interval $|y| \leq \frac{1}{h_j}$, with Lipschitz constant B . Furthermore, the angle between $\bar{\nu}$ and ∂_z at $(1, 0, 0)$ is greater than $\pi/2 - \arctan(2B)$. However, Y_j converges to a vertical half-plane Y_* , and since the convergence is \mathcal{C}^∞ away from the boundary by [6], the angle condition at $(1, 0, 0)$ is preserved in the limit. However

by Lemma 2.2, from the Lipschitz bound on the graph function f_j , we see that $Y_* = \{z = \alpha y + \beta, x > 0\}$ for some α with $|\alpha| \leq B$. This contradicts the angle condition at $(1, 0, 0)$. \square

We also need a slight variant of this.

Lemma 2.4. *Consider a sequence of complete minimal surfaces Y_j , the closures of which pass through the $(0, 0, 0)$. Assume that the subdomains $Y'_{j, B(0, 3)}$ satisfy $\mathcal{E}^{B(0, 3)}(Y_j) \rightarrow 0$, and that $\gamma_j = \partial_\infty Y'_{j, B(0, 3)}$ is a graph $z = f_j(y)$ over the interval $|y| \leq 2$ with $f_j \in C^1$ and $|f'_j(y)| \leq \delta$ for some $\delta > 0$. Then there exists an $\epsilon_0(\delta) > 0$ such that $Y'_{j, B(0, 3)}$ is a graph $z = u_j(x, y)$ over the rectangle $\mathcal{R} := \{0 \leq x \leq \epsilon_0(\delta), |y| \leq 2\}$, and $|\nabla u_j| \leq 2\delta$ on \mathcal{R} .*

Proof. This is proved essentially as before. We pick $\epsilon_0(\delta)$ small enough so that all functions $u_j(x, y)$ with $(x, y) \in [0, \epsilon_0(\delta)] \times [-2, 2]$ whose graphs are portions of upper half-spheres and satisfy $|\partial_y u_j|_{x=0} \leq \delta$ also satisfy $|\nabla u_j(x, y)| \leq \frac{3\delta}{2}$ for $(x, y) \in [0, \epsilon_0(\delta)] \times [-2, 2]$. If the claim were to fail for this $\epsilon_0(\delta)$, we could choose points $P_j \in Y_j$ contained in the portion of Y_j which is graphical over this rectangle where $|\nabla u_j(P_j)| > 2\delta$. Since $Y_j \rightarrow Y_*$ smoothly away from $x = 0$ and Y_* must be a portion of a hemisphere, we must have $x(P_j) \rightarrow 0$. Now dilate by the factor $x(P_j)^{-1}$; this produces a sequence of minimal surfaces \tilde{Y}_j which converge to a vertical half-plane Y_* which meets the xy -plane at a small angle bounded above by $|\arctan(\delta)|$, but such that the corresponding graph functions \tilde{u}_j satisfy $|\nabla \tilde{u}_j| > 2\delta$ at a fixed point $(0, 0, 1)$. However, convergence in this dilated setting is still smooth away from $x = 0$ by [6], so this is a contradiction. \square

3 The ϵ -regularity results: Small energy controls boundary regularity.

3.1 Vanishing energy implies \mathcal{C}^1 boundary convergence.

We first state a key proposition, and then deduce Theorems 1.1, 1.2 from it.

For any $\zeta \in (0, 1]$, consider the (unique) circle C_*^ζ in the yz -plane which is tangent to the y -axis at the origin and whose graph function $f_*^\zeta(y)$ over the interval $[-1, 1]$ satisfies $(f_*^\zeta)'(1) = \zeta$. Pick ζ_0 small enough so that for each $\zeta \in (0, \zeta_0]$ the circle C_*^ζ is contained in the open ball $B(0, \frac{5}{\zeta}) \subset \mathbb{R}^2$.

Proposition 3.1. *Let ζ and ζ_0 be as above. Suppose that Y_j is a sequence of smooth, connected minimal surfaces in $\mathbb{H}^3 \cap B(0, 2)$ with boundaries at infinity $\partial_\infty Y_j = \gamma_j$, and the remaining boundary components on the outer boundary of this half-ball. Assume $\mathcal{E}'(Y_j) \leq M < \infty$. We assume furthermore that:*

- a) *Each γ_j is the graph of a function f_j over $[-1, 1]$, which satisfies $|f_j(y) - f_j(y')| \leq \zeta|y - y'|$ for all $y, y' \in [-1, 1]$ and $f_j(0) = 0$, $f'_j(0) = 0$, $f'_j(1) = \zeta$;*
- b) *$\text{LipRad}_{\gamma_j}^\zeta(P) \geq \frac{2-|P|}{A}$, for some fixed $A > 0$;*
- c) *$\int_{Y_j \cap B(0, 2)} |\tilde{A}_j|^2 d\bar{\mu}_j \leq 1/j$.*

Then $f_j \rightarrow f_^\zeta$ in $\mathcal{C}^1([-1, 1])$.*

In other words, if the energies of a sequence of minimal surfaces converge to zero in some fixed half-ball, and if the boundaries at infinity of these minimal surfaces are uniformly Lipschitz in the qualitative sense above, then these boundaries must converge to a particular circular arc defined by the normalization, and the convergence is actually in \mathcal{C}^1 .

For future reference, we state another proposition which guarantees \mathcal{C}^1 convergence of boundary curves under slightly different assumptions on the boundary

curves. This will be used in the proof of the gain of regularity, which is Theorem 1.4 below.

Proposition 3.2. *Assume that Y_j is a sequence of connected minimal surfaces in $\mathbb{H}^3 \cap B(0, 2)$, with boundaries at infinity $\partial_\infty Y_j = \gamma_j$, and with all other boundaries contained in the outer boundary of the half-ball $B(0, 2)$. Assume $\mathcal{E}'(Y_j) \leq M < \infty$. Assume further that:*

- a) Y_j is the graph of a function $z = u_j(x, y)$ over the half-disc $\{x^2 + y^2 \leq 2, z = 0\}$.
- b) $|\nabla u_j| \leq 2\zeta \leq 1/10$ for $x > 0$ and $f_j(y) := u_j(0, y)$ is a Lipschitz function with Lipschitz constant ζ .
- c) $\mathcal{E}(Y_j) \rightarrow 0$, and Y_j converges to the upper half-disc $\{z = 0, x^2 + y^2 < 2\}$.
- d) All f_j are differentiable at $y = 0$.

Then $\lim_{j \rightarrow \infty} f'_j(0) = 0$.

3.2 Proposition 3.1 implies ϵ -regularity

We now show that Theorems 1.1 and 1.2 can be deduced from Proposition 3.1.

The argument is by contradiction. Assume that for every $j \geq 1$ there exist surfaces $Y_j \in \mathcal{M}$, points $P_j \in \gamma_j := \partial Y_j$ (and radii $R_j \leq 1$ in the context of Theorem 1.1) such that $\mathcal{E}^{B(P_j, R_j)}(Y_j) < \frac{1}{j}$, yet $\text{LipRad}_{\gamma_j}^\zeta(Q_j) < \zeta \frac{R_j - |P_j Q_j|}{10}$ for some $Q_j \in \gamma_j \cap B(P_j, R_j)$. Observe that the points Q_j must lie in the open ball $B(P_j, R_j)$ since $\text{LipRad}^\zeta(\gamma_j) > 0$.

Select a point Z_j in the open ball $B(P_j, R_j)$ so that

$$\inf_Q \frac{\text{LipRad}_{\gamma_j}^\zeta(Q)}{(R_j - |P_j Q|)} = \frac{\text{LipRad}_{\gamma_j}^\zeta(Z_j)}{(R_j - |P_j Z_j|)},$$

and note that this ratio is less than $\zeta/10$. Let $\delta_j := \text{LipRad}_{\gamma_j}^\zeta(Z_j)$. By translation and rotation, assume that $Z_j = 0$ and $T_{Z_j} \gamma_j$ is the y -axis. Now dilate by δ_j^{-1} . Denote the rescaled surface by \tilde{Y}_j and the rescaled boundary curve by $\tilde{\gamma}_j$; note that $|\tilde{\gamma}_j| = 100\pi\delta_j^{-1}$. Thus $\tilde{\gamma}_j$ is a graph $z = f_j(y)$ over $[-1, 1]$, with $f_j(0) = 0$, $f'_j(0) = 0$ and $|f_j(y) - f_j(y')| \leq \zeta|y - y'|$. Moreover, because $[-1, 1]$ is the maximal interval on which the Lipschitz norm of f_j is bounded by ζ , we must have either $|f'_j(-1)| = \zeta$ or $|f'_j(1)| = \zeta$, and to be definite we suppose that $f'_j(1) = \zeta$ for each j .

The translated and rescaled ball \tilde{B}_j contains $B(0, \frac{5}{\zeta})$. Furthermore, by the choice of Z_j and the dilation, we see that there exists an $\eta > 0$ such that for each $P \in \tilde{\gamma}_j \cap B(0, \frac{5}{\zeta})$, $\text{LipRad}_{\tilde{\gamma}_j}^\zeta(P) \geq \eta$.

We claim that $\tilde{\gamma}_j \rightarrow C_*^\zeta$ in \mathcal{C}^1 . Assuming this for the moment, we show that this leads to a contradiction in Theorems 1.1 and 1.2.

For Theorem 1.2, the contradiction is immediate. Indeed, the curves $\tilde{\gamma}_j$ intersect the circle $\partial \tilde{B}_j$, which contradicts the fact that $\tilde{\gamma}_j \rightarrow C_*^\zeta$, which lies strictly in the interior of \tilde{B}_j .

As for Theorem 1.1, let $g_j : [-50\pi\delta_j^{-1}, 50\pi\delta_j^{-1}] \rightarrow \mathbb{R}^2$ parametrize $\tilde{\gamma}_j$ by arclength, so the length along the curve between $g_j(0)$ and $g_j(s)$ is $|s|$; similarly, let $g_* : [-50\pi\delta_j^{-1}, 50\pi\delta_j^{-1}] \rightarrow C_*^\zeta$ be a (multi-covering) arclength parametrization of C_*^ζ .

Observe that $|C_*^\zeta| = 2\pi R_\zeta$, with $R_\zeta = \sqrt{1 + \frac{1}{\zeta^2}}$. Our claim gives that $g_j(s) \rightarrow g_*(s)$ in $\mathcal{C}^1([-\pi R_\zeta, \pi R_\zeta])$, so in particular,

$$\lim_{j \rightarrow \infty} g_j(-\pi R_\zeta) = \lim_{j \rightarrow \infty} g_j(\pi R_\zeta), \quad \lim_{j \rightarrow \infty} g'_j(-\pi R_\zeta) = - \lim_{j \rightarrow \infty} g'_j(\pi R_\zeta).$$

This shows that $\lim_{j \rightarrow \infty} |\tilde{\gamma}_j| = |C_*^\zeta|$. On the other hand, we know that $|\tilde{\gamma}_j| = 100\pi\delta_j^{-1}$, which is impossible since $|C_*^\zeta| = 2\pi R_\zeta$.

Proof that Proposition 3.1 implies $\gamma_j \rightarrow C_^\zeta$ in \mathcal{C}^1 :* Let $2\zeta'$ be the length of the arc in C_*^ζ which is a graph over the interval $y \in [-1, 1]$. By Proposition 3.1, $g_j(s) \rightarrow g_*(s)$ for $s \in [-\zeta', \zeta']$. Let $M > 0$ be the largest number in $[0, R_\zeta]$ such that $g_j(s) \rightarrow g_*(s)$ in $\mathcal{C}^1([-M, M])$. We must prove that $M = R_\zeta$, and moreover, for any small $\epsilon > 0$ and j sufficiently large, that there exists an $\epsilon_j > 0$ with $\lim_{j \rightarrow \infty} \epsilon_j = \epsilon$ and $g_j(-R_\zeta - \epsilon_j) = g_j(R_\zeta - \epsilon)$. The first claim ensures that $\gamma_j([-R_\zeta, R_\zeta]) \rightarrow C_*^\zeta$, while the second implies that $\gamma_j(s)$ closes up on a small extension of the interval $[-R_\zeta, R_\zeta]$.

The first part is proved by contradiction: Assume $M < R_\zeta$, and consider the pairs $(g_j(M), g'_j(M))$. These converge to $(g_*(M), g'_*(M))$, so for j large they lie in the open set \mathcal{U} where LipRad is bounded below by some $\eta > 0$. Now let ℓ_j be the tangent line to $\tilde{\gamma}_j$ at $g_j(M)$. Consider the intervals of length η centered at $g_j(M)$ on each ℓ_j . After translation, rotation and dilation by the factor η^{-1} , the rescaled $\tilde{\gamma}_j$ can be written as the graphs of functions ϕ_j on $[-1, 1]$. Applying Proposition 3.1 to these functions, we see that $\phi_j \rightarrow f_*$ in \mathcal{C}^1 . Hence $g_j \rightarrow g_*$ on a larger interval $[-M', M']$, which contradicts the maximality of M .

As for the second part of the claim, note that the argument above shows that for $|\tau| \leq \eta$ we have

$$\lim_{j \rightarrow \infty} (g_j(-R_\zeta - \tau), g'_j(-R_\zeta - \tau)) = \lim_{j \rightarrow \infty} (g_j(R_\zeta - \tau), -g'_j(R_\zeta - \tau)),$$

because of the lower bound $\text{LipRad}^\zeta \geq \eta$ and the \mathcal{C}^1 convergence of the g_j on $[-R_\zeta - \tau, -R_\zeta + \tau]$ to an arc of C_*^ζ .

Now, assume that for some fixed $\epsilon > 0$, there exists a subsequence in j such that $g_j(-R_\zeta - \epsilon) \neq g_j(R_\zeta - s)$ for any $s \in (0, 2\epsilon)$. In particular this says that $g_j(t)$ does not “close up” for $t \leq -R_\zeta$ and $t \geq R_\zeta$.

This gives a sequence of values $\tau_j \in [-50\pi\delta_j^{-1}, -R_\zeta] \cup [R_\zeta, 50\pi\delta_j^{-1}]$ such that $\tau_j \rightarrow \tau_*$, $g_j(-\tau_j) \rightarrow P$, with $P \in C_*^\zeta$, yet $g'_j(\tau_j) \rightarrow T_*$ for some vector T_* which is transverse to the tangent vector T of C_*^ζ at P . However, if j is large enough, then $g_j(\tau_j) \in \mathcal{U}$, and hence $\text{LipRad}_{\gamma_j}^\zeta(g_j(\tau_j)) \geq \eta > 0$. But this implies that γ_j must *self-intersect* near P , which contradicts that the boundary curves are embedded. \square

3.3 An overview of the strategy

In the next two sections, we prove Propositions 3.1 and 3.2. In a nutshell, both results show, in slightly different settings, that if the energies of portions of minimal surfaces $Y_j \subset \mathbb{H}^3$ converge to zero, then $\gamma_j = \partial_\infty Y_j$ must converge *in the \mathcal{C}^1 norm* to the boundary curve γ_* of a totally geodesic surface Y_* . We stress that the convergence of Y_j to Y_* is \mathcal{C}^∞ away from $\{x = 0\}$; the novelty here is the \mathcal{C}^1 convergence at the boundary.

Since the argument has several steps, we now provide a moderately detailed outline of the strategy. If the results were false, we could find a sequence of minimal surfaces Y_j satisfying the hypotheses but for which the \mathcal{C}^1 convergence fails at some boundary point. Thus, having written the boundary curves graphically, we assume that there exists $y_0 \in [0, 1]$ such that $\lim_{j \rightarrow \infty} f'_j(y_0) = b_1 \neq b_2 = f'_*(y_0)$. Because the local energy converges to zero, the limit Y_* is totally geodesic, and the convergence is \mathcal{C}^∞ away from $\{x = 0\}$. Furthermore, at $\{x = 0\}$, $f_j \rightarrow f_*$ in \mathcal{C}^α , where the graph of f_* is a circular arc.

Compose with a suitable sequence of rotations, reflections and inversions so that we can assume that $(y_0, f_j(y_0)) = (0, 0)$ and (maintaining the names of all surfaces and curves) that Y_* is a portion of the vertical plane $\{z = 0\}$. By assumption b) of Proposition 3.1, each γ_j is the graph of a functions f_j defined on a fixed interval $[-1, 1]$, and the limiting curve γ_* is the graph of $f_* = 0$ on this same interval. The hypothesis is that $\lim_{j \rightarrow \infty} f'_j(0) = a > 0$, although $f'_*(0) = 0$.

The argument proceeds in two steps. We first show that there exists a sequence of hyperbolic isometries φ_j such that the surfaces $\varphi_j(Y_j)$ satisfy all the assumptions

of Propositions 3.1 and 3.2 (including the jump in the limit of the first derivatives), but so that some fixed portion of $\varphi_j(Y_j)$ is covered by isothermal coordinates, the associated conformal factor of which is uniformly bounded. This construction relies crucially on ideas in [8], many of which go back to the influential paper [19]. The work here will involve modifying some arguments in [8], which is possible because of some special features of our setting, to ensure that the jump in first derivative has a fixed size $\alpha - \beta > \frac{\alpha}{2}$.

However, we then use particular properties of these isothermal coordinate systems to prove that no such jump in the limit of the first derivatives can occur. Writing $\varphi_j(Y_j)$ as the graphs of functions u_j , and denoting the isothermal coordinates by (q_j, w_j) , the idea is to control $\partial_{w_j} u_j|_{(0,0)}$ using that $\partial_{w_j} u_j \rightarrow 0$ as $j \rightarrow \infty$ uniformly along $\{x = 1\}$. The relationship between these derivatives at $x = 0$ and $x = 1$ is obtained using two integrals, the first of the mixed component of the second fundamental form of $\varphi_j(Y_j)$ with respect to the Euclidean metric, and second depends on a derivative of the conformal factor. We show that these integrals are bounded in terms of $\mathcal{E}(\varphi_j(Y_j))$ and hence converge to 0. The estimate of the first integral uses the superharmonicity of $x^4 A_j(\partial_{q_j}, \partial_{w_j})^2$, which in turn follows from an old result of Calabi and Chern that $A(\partial_q, \partial_w)$ is harmonic, and then a monotonicity formula for superharmonic functions, to obtain rather strong pointwise bounds on $\bar{A}(\partial_q, \partial_w)$. These show that the first integral tends to 0 as $j \rightarrow \infty$. Control of the second integral follows from interpreting it as one term in a flux formula whose interior term is controlled by $\mathcal{E}(\varphi_j(Y_j))$.

Remark 3.1. *The jump of the first derivative of γ_j can also be described in terms of the Euclidean coordinate function z restricted to the surface Y_j . Indeed, the jump condition is the same as*

$$|\lim_{j \rightarrow \infty} \bar{\nu}_j(z) - \bar{\nu}_*(z)| = \alpha > 0, \quad (3.1)$$

where $\bar{\nu}_j$ and $\bar{\nu}_*$ are the Euclidean unit tangent vectors to $\partial_\infty Y_j$ and $\partial_\infty Y_*$ at $(0, 0, 0)$.

4 Uniform isothermal parametrizations

The next step is to choose a sequence of hyperbolic isometries φ_j which map the surfaces Y_j to a new sequence of surfaces which satisfy the assumptions of our propositions (in particular they converge to a vertical half-plane) but such that some fixed portions of these rescaled surfaces admit isothermal coordinates (q_j, w_j) , the conformal factors of which are uniformly bounded in \mathcal{C}^0 , $W^{2,1}$ and $W^{1,2}$. We must ensure that the transformed surfaces still exhibit a jump in first derivative at the origin.

Let us put this into context. In their well-known paper [19] (see also [11]), Müller and Sverak show that surfaces of finite total curvature in \mathbb{R}^3 admit isothermal parametrizations with bounded conformal factors. This argument was extended by De Lellis and Müller [8] to obtain a uniformization of spheres $\Sigma \subset \mathbb{R}^3$, proving in particular that if $\int_\Sigma |\bar{A}|^2 d\mu < 8\pi$, then one can find a conformal map $\Psi : S^2 \rightarrow \Sigma$ for which the conformal factor is controlled by this energy. In order to deal with the noncompactness of the conformal group, which ensures the existence of such maps with conformal factor having arbitrarily large supremum, they impose a normalization in the form of a balancing condition for the total curvature restricted to certain hemispheres. Our argument below follows this idea, with one key difference: we obtain this balancing not intrinsically by precomposing with conformal maps of S^2 , but extrinsically, by post-composing by Möbius transformations of \mathbb{R}^3 . This is necessary for ensuring that the jump in first derivative still occurs in these isothermal coordinates. We note that this extrinsic balancing may not be possible for arbitrary spheres $\Sigma \subset \mathbb{R}^3$ close to S^2 , but we are able to set things up so that our surfaces already have two reflection symmetries and only one extra balancing condition needs to be attained.

This construction is used in the proofs of Propositions 3.1 and 3.2 in slightly different settings, so we prove the present result in two different settings as well. These involve different hypotheses on the boundary curves $\gamma_j = \partial_\infty Y_j$, which we always assume pass through the origin. The γ_j are \mathcal{C}^1 or Lipschitz, respectively, with uniform control on the norms, and in the second setting, we assume that γ_j is differentiable at the origin. Let us now describe these more carefully. In the following, and throughout the rest of this section, we write

$$D_+(a) = \{(x, y, 0) : x^2 + y^2 \leq a^2, x \geq 0\}, \quad D(a) = \{(x, y, 0) : x^2 + y^2 \leq a^2\}$$

for the half-disk or disk of radius a in the vertical plane $\{z = 0\}$.

Setting 1: Y_j is a sequence of incomplete minimal surfaces, where each Y_j is a horizontal graph $z = u_j(x, y)$ over $D_+(3)$ with $u_j \in \mathcal{C}^2$, $\|u_j\|_{W^{2,2}} \leq M < \infty$, $u_j(0, 0) = 0$ and $\partial_y u_j(0, 0) = \alpha > 0$. We assume that $|\nabla u_j|_{\overline{g}} \leq \zeta \leq 1/20$, and finally that $\mathcal{E}(Y_j) \leq \mu < 2\pi$ and $Y_j \rightarrow Y_* = \{(x, y, 0) : x \geq 0\}$.

Setting 2: Y_j is a sequence of incomplete minimal surfaces which are again horizontal graphs $z = u_j(x, y)$ over $D_+(3)$ with $u_j(0, 0) = 0$ and $u_j \in W^{2,2}$, $\|u_j\|_{W^{2,2}} \leq M < \infty$, and $u_j \in \mathcal{C}^2$ away from $\{x = 0\}$. We assume that $(0, 0)$ is a point of differentiability for u_j and $\partial_y u_j(0, 0) = \alpha > 0$. We also assume that $y \mapsto u_j(0, y)$ is Lipschitz with constant $\zeta \leq 1/20$, and furthermore, $|\nabla u_j| \leq 2\zeta$ for $x > 0$. Finally, suppose that $\mathcal{E}(Y_j) \leq 2\pi$ and $Y_j \rightarrow Y_* = \{(x, y, 0) : x \geq 0\}$.

Recalling that α is the jump in the derivative, choose any number β with $0 < \beta \ll \alpha$. Consider the straight line $\ell_\beta = \{z = \beta y\}$ in the horizontal plane $\{x = 0\}$. Since the curves γ_j converge to a segment in the y -axis containing the subinterval $[-1, 1]$, then for j large, there must exist values $-1 < y_j^- < 0 < y_j^+ < 1$ such that the two points $F_j^\pm = (0, y_j^\pm, u_j(y_j^\pm))$ both lie on the line ℓ_β . We assume that y_j^+ is chosen as large as possible in the interval $(0, 1)$, and similarly for y_j^- . Since $\gamma_j = \text{Graph}(u_j|_{x=0})$ converges to the line $\ell_0 = \{z = 0\}$, it is necessarily the case that $|F_j^\pm| \rightarrow 0$. Let $R_{-\beta}$ denote the rotation of the yz -plane by the small negative angle which sends ℓ_β to ℓ_0 ; thus $R_{-\beta}(F_j^\pm) = (\pm|F_j^\pm|, 0)$.

Suppose, to be definite, that $|F_j^+| \geq |F_j^-|$. Dilating the entire surface by the factor $|F_j^+|^{-1}$ pushes the point F_j^+ to $(1, 0)$. The key observation is that this dilation of $R_{-\beta}Y_j$ converges to a vertical plane (since it must be totally geodesic and graphical over $\{z = 0\}$), and since this plane contains the two points $(0, 0)$ and $(1, 0)$, it *must* be $\{z = 0, x \geq 0\}$. This holds even though, before dilating, the sequence $R_{-\beta}Y_j$ converges to the vertical plane $\{y = -\beta z, x \geq 0\}$. Denote this dilated, rotated surface by $\tilde{R}_{-\beta}(Y_j)$. Note also that given the upper bound $\int_{Y_j} |\bar{A}|^2 d\bar{\mu} \leq M$ (which follows from the $W^{2,2}$ bounds in our assumption), it follows that given any $\gamma > 0$ and any fixed numbers $0 < \beta_- < \beta_+ \ll \alpha$ then for j large enough there exists some $\beta_- < \beta_j < \beta_+$ such that

$$\int_{\tilde{R}_{-\beta_j}(Y_j) \cap \{1/4 \leq x^2 + y^2 + z^2 \leq 9\}} |\bar{A}|^2 d\bar{\mu} \leq \gamma.$$

Remark 4.1. *By this observation, it follows that we can pick a sequence $\beta_j, 0 < \beta_- \leq \beta_j < \beta_+ \ll \alpha$ such that:*

$$\int_{\tilde{R}_{-\beta_j}(Y_j) \cap \{1/4 \leq x^2 + y^2 + z^2 \leq 9\}} |\bar{A}|^2 d\bar{\mu} = o(1). \quad (4.1)$$

We make this choice hereafter.

For simplicity, we now reset the notation and write the rotated dilated surfaces as Y_j , with boundary curves γ_j , graph functions u_j , etc. Also, $o(1)$ denotes a sequence of numbers which converges to 0 as $j \rightarrow \infty$.

Lemma 4.1. *Consider a sequence of incomplete minimal surfaces Y_j which are graphs $z = u_j(x, y)$ over $D_+(3)$ with $|\nabla u_j| \leq 2\zeta$, $\text{Lip}(u_j|_{x=0}) \leq \zeta$, $8\mathcal{E}(Y_j) \leq \pi$, $\int_{Y_j \cap \{1/4 \leq x^2 + y^2 + z^2 \leq 9\}} |\bar{A}|^2 d\bar{\mu} \rightarrow 0$, $u_j(0, 0) = 0$, $u_j(0, 1) = 0$ and $u_j \rightarrow 0$, where the convergence is in C^∞ away from $\{x = 0\}$ and in $C^{0, \alpha}$ up to $x = 0$. Assume further that there is a jump in the first derivative at the origin:*

$$\lim_{j \rightarrow \infty} \partial_y u_j(0, 0) - \partial_y u_*(0, 0) \geq \alpha - 2\beta_j > \frac{1}{2}\alpha. \quad (4.2)$$

Then, there exist Möbius transformations φ_j and open sets $\mathcal{U}_j \subset Y_j$ such that $\tilde{Y}_j = \varphi_j(\mathcal{U}_j)$ are graphs $z = \tilde{u}_j(x, y)$ over $D_+(2)$ with $|\nabla \tilde{u}_j| \leq 4\zeta$, $\tilde{u}_j(0, 0) = 0$, $\tilde{u}_j(0, 1) = 0$ and

$$\lim_{j \rightarrow \infty} \partial_y \tilde{u}_j(0, 0) - \partial_y \tilde{u}_*(0, 0) \geq \alpha - 2\beta_j > \frac{1}{2}\alpha. \quad (4.3)$$

Furthermore, there exist isothermal coordinate charts (q_j, w_j) centered at the origin and covering the region $\varphi_j(\mathcal{U}_j)$ with $q_j = 0$ along γ_j , such that $C'^{-1} \leq |\nabla q_j|_{\bar{g}} \leq C'$ for some constant $C' > 0$ which depends only on $\sup_j \mathcal{E}(Y_j)$. Finally, the conformal factor ϕ_j associated with the coordinates q_j, w_j satisfies the estimates:

$$\|\phi_j\|_{C^0(\varphi_j(\mathcal{U}_j))} + \|\phi_j\|_{W^{1,2}(\varphi_j(\mathcal{U}_j))} + \|\phi_j\|_{W^{2,1}(\varphi_j(\mathcal{U}_j))} \leq C \int_{\varphi_j(\mathcal{U}_j)} |\bar{A}_j|^2 d\bar{\mu} + o(1). \quad (4.4)$$

Remark 4.2. *For future reference, note that we actually prove that the surfaces $\varphi_j(\mathcal{U}_j)$ are subregions of complete, smooth graphical surfaces Y_j^b in \mathbb{R}^3 which are reflection-symmetric across $\{x = 0\}$. If $u_j^b(x, y)$ is the graph function of Y_j^b , then the (distorted) annular regions $\{(x, y, u_j^b(x, y)), 2 \leq \sqrt{x^2 + y^2} \leq 4\}$ of these larger surfaces are not minimal with respect to the hyperbolic metric. On the other hand, $u_j^b(x, y) = 0$ for $\sqrt{x^2 + y^2} \geq 5$; we denote this portion of Y_j^b by Y_j^\sharp . The isothermal coordinates q_j, w_j cover the entire surface Y_j^b , and the associated conformal factor ϕ_j satisfies (4.4) on all of Y_j^b and $\phi_j \rightarrow 0$ as $\sqrt{x^2 + y^2} \rightarrow \infty$. In the first setting above, $\mathcal{E}[Y_j^b] \rightarrow 0$.*

Remark 4.3. *The pointwise bound on $|\nabla q_j|_{\bar{g}}$ follows from the C^0 bounds on ϕ_j . Indeed, dropping the subscript j momentarily, we have*

$$\bar{g} = e^{2\phi}(dq^2 + dw^2) = (1 + (u_x)^2)dx^2 + 2u_x u_y dx dy + (1 + (u_y)^2)dy^2, \quad (4.5)$$

so in particular

$$\bar{g}(\partial_x, \partial_x) + \bar{g}(\partial_y, \partial_y) = 2 + u_x^2 + u_y^2 = e^{2\phi}(|\partial_x q|^2 + |\partial_y q|^2 + |\partial_x w|^2 + |\partial_y w|^2).$$

Using that $|u_x|, |u_y| \leq 1/10$ and that $|dq|_{\bar{g}} = |dw|_{\bar{g}}$, $dq \perp dw$, the equivalence of pointwise bounds on ϕ_j and $|\nabla q_j|_{\bar{g}}$ follows directly.

Proof. The main work is to establish the existence of the isothermal parametrization, so we concentrate on this; properties of the graphical representation are derived at the end.

As described at the beginning of this section, we wish to apply a theorem of De Lellis and Müller [8]. There are a few technical points we must deal with before we can apply this result. First, the surfaces in the statement of this theorem are local, so we must extend them to closed topological spheres. This is done by first reflecting each Y_j across the horizontal plane, then extending the resulting perturbed disk to a surface which agrees with the vertical plane $\{z = 0\}$ outside a large ball, then stereographically projecting. Next, to obtain a balanced configuration as described above (and in more detail below), we can arrange for the perturbed S^2 to have two reflection symmetries immediately, then obtain the third balancing condition by composing with an appropriate Möbius transformation in \mathbb{R}^3 .

The upshot of all of this is that we obtain isothermal coordinates (q, w) which still detect the jump in the first derivative, and with $0 < C_1 \leq |\nabla q|, |\nabla w| \leq C_2$.

Reflection: We first reflect Y_j across the horizontal plane to obtain a surface Y'_j in \mathbb{R}^3 invariant with respect to the vertical reflection $x \mapsto -x$. The doubled surface is graphical over $D(3)$, and has graph function $\tilde{u}_j \in W^{2,2}(D(3))$. This is straightforward to check using Lemma 2.1. We change notation, denoting the doubled surface Y'_j by Y_j again.

Extension: We now claim that the doubled incomplete surface Y_j can be extended to a complete surface Y_j^{ext} which is a graph over the entire vertical plane $\{z = 0\}$ with graph function u_j^{ext} which vanishes when $x^2 + y^2 \geq 25$ and also satisfies

$$\mathcal{E}(Y_j^{\text{ext}}) \leq 2\mathcal{E}(Y_j) + o(1). \quad (4.6)$$

Notice that Y_j^{ext} is no longer minimal in the transition annulus $4 \leq x^2 + y^2 \leq 9$.

Since this construction is a bit lengthy, we defer it to §4 below, so let us grant it for the time being.

Mollification: We will eventually apply the uniformization theorem, so we wish to smooth out our surfaces via a mollification. We will check convergence properties of the uniformization map for the mollified surfaces momentarily. The mollification is standard: Choose $\psi \in C_c^\infty(\mathbb{R}^2)$ with $\int \psi = 1$ and set $\psi_\epsilon(x, y) := \epsilon^{-2}\psi(x/\epsilon, y/\epsilon)$. Then let $Y_{j,\epsilon}^{\text{ext}}$ be the graph of the function

$$u_{j,\epsilon}^{\text{ext}}(x, y) = u_j^{\text{ext}} * \psi_\epsilon(x, y).$$

It is standard that for each $\epsilon \in (0, 1]$, $u_{j,\epsilon}^{\text{ext}} \in C^\infty$ and $\|u_{j,\epsilon}^{\text{ext}}\|_{W^{2,2}} \rightarrow \|u_j^{\text{ext}}\|_{W^{2,2}}$. Furthermore, given a priori C^1 or Lipschitz bounds on u_j^{ext} , there are uniform C^1 bounds (independent of ϵ and j) on $u_{j,\epsilon}^{\text{ext}}$. In particular,

$$\int_{Y_{j,\epsilon}^{\text{ext}}} |\bar{A}_{j,\epsilon}|^2 d\bar{\mu} \rightarrow \int_{Y_j^{\text{ext}}} |\bar{A}_j|^2 d\bar{\mu} \quad (4.7)$$

as $\epsilon \rightarrow 0$.

Patching into a sphere with symmetries: Let I be the Möbius transformation of \mathbb{R}^3 which maps the plane $\{z = 0\}$ to the sphere $S_1(0)$, normalized by requiring that $I((0, 0, 0)) = (0, 0, -1) := S$ (the south pole), $I(\infty) = (0, 0, 1) := N$ (the north pole), and so that I carries the y -axis to the great circle $C := \{z = 0\} \cap S_1(0)$ minus N . This map is fully determined by requiring that the image of the disc $D(2)$ equals the spherical cap in $S_1(0)$ of radius $1/10$ centered at S .

Now let $Y'_{j,\epsilon} := I(Y_{j,\epsilon} \cup \{\infty\})$; this is a slightly distorted sphere, where the distortion is localized near S . This surface has one reflection symmetry, across the plane $x = 0$, corresponding to the original vertical reflection symmetry. By construction, $Y'_{j,\epsilon}$ coincides with standard unit sphere in a neighbourhood of the closed northern hemisphere $S_1(0) \cap \{z \geq 0\}$, so we can form a new surface $\hat{Y}_{j,\epsilon}$ by discarding this northern hemisphere and replacing it with a reflection of the southern hemisphere of $Y'_{j,\epsilon}$. The surface we obtain this way is smooth near its intersection with the horizontal plane $\{z = 0\}$, and has two reflection symmetries, one across the plane $\{z = 0\}$ and the other across $\{x = 0\}$.

Now, consider a uniformizing map $\psi_{j,\epsilon}$ from the round sphere to $(\hat{Y}_{j,\epsilon}, \bar{g}_{j,\epsilon})$, where $\bar{g}_{j,\epsilon}$ is the metric on $\hat{Y}_{j,\epsilon}$ induced from \mathbb{R}^3 . Because of the symmetries of $\hat{Y}_{j,\epsilon}$, the map $\psi_{j,\epsilon}$ can be chosen to be reflection-symmetric across the xy - or yz -planes. We can also assume that the conformal maps $\psi_{j,\epsilon}$ converge in $W^{1,\infty}$ to a conformal map ψ_j as $j \rightarrow \infty$. This is done using an inversion, coupled with the graphicality property and [19]. Indeed, first consider the inversion \tilde{I} of \mathbb{R}^3 which sends $(1, 0, 0)$ to infinity and fixes $(-1, 0, 0)$. It follows (by the same argument as in the graphicality discussion

below) that $\tilde{I}(\hat{Y}_{j,\epsilon})$ is a graph over the plane $\{x = 0\}$. We thus obtain a graph function $x = f_{j,\epsilon}(y, z)$ such that $f_{j,\epsilon} = 0$ for $y^2 + z^2 \geq M$, where M can be chosen independent of ϵ . By the first paragraph in the proof of Theorem 5.2 in [19], there exists a 1-parameter family of smooth conformal parametrizations $\hat{\psi}_{j,\epsilon} : \mathbb{R}^2 \rightarrow \tilde{I}(\hat{Y}_{j,\epsilon})$ with $\hat{\psi}_{j,\epsilon} \rightarrow 0$ at ∞ , such that $\hat{\psi}_{j,\epsilon}$ converges in $W^{1,\infty}$ to a parametrization $\hat{\psi}_j : \mathbb{R}^2 \rightarrow \tilde{I}(\hat{Y}_j)$. Since the conformal factor $\hat{\phi}_{j,\epsilon}$ associated to $\hat{\psi}_{j,\epsilon}(y, z)$ is harmonic with respect to the flat metric on $y^2 + z^2 \geq M$, standard asymptotics results for harmonic functions on exterior domains give that $|\partial^2 \hat{\phi}_{j,\epsilon}| = o((x^2 + y^2)^{-1})$, uniformly in ϵ for j fixed. This implies that the conformal maps $\psi_{j,\epsilon} := \tilde{I}^{-1} \circ \hat{\psi}_{j,\epsilon} \circ \tilde{I} : S^2 \rightarrow \hat{Y}_{j,\epsilon}$ converge in $W^{1,\infty}$ to a conformal map $\psi_j : S^2 \rightarrow \hat{Y}_j$. We consider these $\psi_{j,\epsilon}$ hereafter.

Balancing the total curvature: This is the key step in the derivation of our estimates. As already mentioned, [8] requires that we find a conformal map $\psi_{j,\epsilon} : S^2 \rightarrow \hat{Y}_{j,\epsilon}$ which satisfies three separate balancing conditions: if $H_{\pm,b}$, $b = x, y, z$, denote the three pairs of hemispheres in S^2 centered along the coordinate axes with the same labels, then we demand that

$$\int_{\psi(H_{+,b})} |\bar{A}_{j,\epsilon}|^2 d\bar{\mu} = \int_{\psi(H_{-,b})} |\bar{A}_{j,\epsilon}|^2 d\bar{\mu}, \quad (4.8)$$

for all three choices of b . Our conformal map $\psi_{j,\epsilon} : S^2 \rightarrow \hat{Y}_{j,\epsilon}$ clearly respects the two reflection symmetries of this target surface, and for this map, (4.8) is satisfied for $b = x$ and z .

To obtain the third balancing condition, we modify $\psi_{j,\epsilon}$ by composing it with a Möbius transformation M_t which is a hyperbolic dilation with source $(0, -1, 0)$ and sink $(0, 1, 0)$. Notice that $\psi_{j,\epsilon}$ fixes these two points already. Each M_t preserves the unit sphere $S_1(0)$. Now consider the family of surfaces $M_t(\hat{Y}_{j,\epsilon})$. These all have the two original reflection symmetries simply because M_t respects those reflections. We claim that for each j, ϵ , there exists a unique $t_{j,\epsilon}$ such that

$$\int_{M_{t_{j,\epsilon}} \circ \psi_{j,\epsilon}(H_{+,y})} |\bar{A}_{j,\epsilon}|^2 d\bar{\mu} = \int_{M_{t_{j,\epsilon}} \circ \psi_{j,\epsilon}(H_{-,y})} |\bar{A}_{j,\epsilon}|^2 d\bar{\mu}.$$

To prove this, first note that if Σ is a smooth closed surface in \mathbb{R}^3 diffeomorphic to the sphere, then $|\bar{A}_\Sigma|^2 = 2|\bar{\bar{A}}_\Sigma|^2 + 2K_\Sigma$, where K_Σ is the Gauss curvature of Σ , hence

$$\int_\Sigma |\bar{A}_\Sigma|^2 d\bar{\mu} = 2 \int_\Sigma |\bar{\bar{A}}_\Sigma|^2 d\bar{\mu} + 8\pi.$$

The two terms on the right are conformally invariant, and hence preserved if we apply any one of the maps M_t to Σ ; therefore so is the left side. We also remark here that by construction, $8\pi \leq \int_{\hat{Y}_{j,\epsilon}} |\bar{A}_{j,\epsilon}|^2 d\bar{\mu} \leq 10\pi$ for j large. Now, $\hat{Y}_{j,\epsilon}$ agrees with the standard round sphere in a small neighbourhood around the two points $W = (0, -1, 0)$ and $E = (0, 1, 0)$. Recall also that $\psi_{j,\epsilon} : S^2 \rightarrow \hat{Y}_{j,\epsilon}$ maps the points $\mathbf{W} = (0, -1, 0)$, $\mathbf{E} = (0, 1, 0)$ in S^2 to W , E in $\hat{Y}_{j,\epsilon}$. Choose a small disk centered at \mathbf{W} ; its image under $\psi_{j,\epsilon}$ will then be a small cap around W . Now, the image of this cap under M_t with $t \gg \infty$ is a large spherical cap which covers almost the entire sphere except a small neighbourhood of E , hence the integral of $|\bar{A}|^2$ over this region is very close to 8π . Analogously, at $t \gg -\infty$ its image is a tiny spherical cap centered at W , hence the integral of $|\bar{A}|^2$ over this region is very close to 0. Since the total energy of $\hat{Y}_{j,\epsilon}$ is just slightly larger than 8π , this gives the existence of the value $t_{j,\epsilon}$, as claimed. Since $\int_K |\bar{A}_{j,\epsilon}|^2 \rightarrow \int_K |\bar{A}_j|^2$ and the conformal maps $\psi_{j,\epsilon} \rightarrow \psi_j$ uniformly in all of S^2 as $\epsilon \rightarrow 0$ for each fixed j , where K is any fixed closed subset, this argument shows that there is a bound $|t_{j,\epsilon}| \leq T_j$ which is uniform in ϵ .

The preferred conformal transformation is now given by

$$\Psi_{j,\epsilon} = M_{t_{j,\epsilon}} \circ \psi_{j,\epsilon} : S_1(0) = S^2 \longrightarrow M_{t_{j,\epsilon}}(\hat{Y}_{j,\epsilon}).$$

If \bar{g}_0 is the standard round metric on $S_1(0)$ and $\hat{g}_{j,\epsilon}$ is the metric on $M_{t_j,\epsilon}(\hat{Y}_{j,\epsilon})$ induced from the Euclidean metric in \mathbb{R}^3 , then define $\phi_{j,\epsilon}$ by

$$\Psi_j^* \hat{g}_{j,\epsilon} = e^{2\phi_{j,\epsilon}} \bar{g}_0.$$

Proposition 3.2 and Theorem 3.3 in [8] and the $W^{2,1}$ estimates in their proof now give that

$$\|\phi_{j,\epsilon}\|_{C^0} + \|\phi_{j,\epsilon}\|_{W^{1,2}} + \|\phi_{j,\epsilon}\|_{W^{2,1}} \leq C \int_{M_{t_j,\epsilon}(\hat{Y}_{j,\epsilon})} |\mathring{A}_{j,\epsilon}|^2 d\bar{\mu}. \quad (4.9)$$

As a brief hint of the idea of the proof of this fact, $w_{j,\epsilon}$ is a solution of the semilinear elliptic PDE, $\Delta_{g_0} w_{j,\epsilon} = 1 - \hat{K}_{j,\epsilon} e^{2w_{j,\epsilon}}$, where $\hat{K}_{j,\epsilon}$ is the Gauss curvature function on $M_{t_j,\epsilon}(\hat{Y}_{j,\epsilon})$. The main term $\hat{K}_{j,\epsilon} e^{2w_{j,\epsilon}}$ on the right has a ‘determinant structure’, since it can be expressed via the pullback of the area form on S^2 by the Gauss map. After appropriately modified stereographic projections (localized to be trivial in certain regions of the sphere) this allows one to conclude that the right hand side lies in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$, and from there the estimates follows from some important and well-known theorems in harmonic analysis. We refer to [8] and for further details.

We now pass to the limit as $\epsilon \rightarrow 0$. Because $t_{j,\epsilon}$ is bounded uniformly in ϵ for each j we can pass to a subsequence (in ϵ) and assume that $t_{j,\epsilon} \rightarrow t_j$; with no loss of generality, and possibly taking a reflection, we assume that $t_j \geq 0$ for all j . Following the argument in Müller-Sverak [19], we obtain limiting functions $\psi_{j,\epsilon} \rightarrow \psi_j$ and $\phi_{j,\epsilon} \rightarrow \phi_j$, where

$$\|\phi_j\|_{C^0} + \|\phi_j\|_{W^{1,2}} + \|\phi_j\|_{W^{2,1}} \leq C \int_{M_{t_j}(\hat{Y}_j)} |\mathring{A}_j|^2 d\bar{\mu} \leq 4C\mathcal{E}(Y_j) + o(1). \quad (4.10)$$

The second inequality here follows from (4.6), (4.7) and the assumption that the total curvature in an annular region converges to zero.

Undoing the stereographic projection: For brevity, set $\bar{Y}_j = M_{t_j}(\hat{Y}_j)$. The points P_j and Q_j which correspond to the points of intersection $(0, 0, 0)$ and $(0, 1, 0)$ of the boundary curve and the y -axis in the original (dilated and rotated) surface Y_j correspond under $M_{t_j} \circ \psi_j$ to new points, which we still label as P_j and Q_j , on $\bar{Y}_j \cap \{x = 0\}$. These lie on the circle $S_1(0) \cap \{x = 0\}$.

We have denoted by \mathbf{E} and \mathbf{W} the points $(0, 1, 0)$ and $(0, -1, 0)$ in S^2 , respectively; the corresponding points $(0, \pm 1, 0)$ in \bar{Y}_j have been labelled E and W . Let B be the Möbius transformation of \mathbb{R}^3 which induces the stereographic projection from W onto the plane $\{y = 1\}$. Let B^\sharp be the stereographic projection from $S^2 \setminus \mathbf{W}$ to \mathbb{R}^2 , which sends \mathbf{E} to $0 \in \mathbb{R}^2$ and \mathbf{W} to ∞ .

If $\tilde{Y}_j := B(\bar{Y}_j)$, then define

$$\tilde{\Psi}_j : \mathbb{R}^2 \rightarrow \tilde{Y}_j, \quad \tilde{\Psi}_j := B \circ \Psi_j \circ (B^\sharp)^{-1}.$$

This map is conformal, and determines the functions q_j, w_j as the push-forwards of the flat coordinates q, w on \mathbb{R}^2 , so that $\tilde{\Psi}_j^*(\bar{g}) = e^{2\phi_j}(dq_j^2 + dw_j^2)$. The points P_j, Q_j are mapped to points \tilde{P}_j, \tilde{Q}_j on the line $\{y = 1, x = 0\}$

If $B(\tilde{P}_j, |\tilde{P}_j \tilde{Q}_j|)$ denotes the ball in \mathbb{R}^3 centered at \tilde{P}_j and with radius $|\tilde{P}_j \tilde{Q}_j|$, then on $\tilde{Y}_j \cap B(\tilde{P}_j, |\tilde{P}_j \tilde{Q}_j|)$, the conformal factor $\tilde{\phi}_j$ is obtained by adding to ϕ_j a smooth function w_j which is *a priori* bounded since it the conformal factor for a stereographic projection restricted to the domain $\tilde{Y}_j \cap B(\tilde{P}_j, |\tilde{P}_j \tilde{Q}_j|)$ which is uniformly bounded away from the point that is mapped to infinity. It is not hard to see that since $\tilde{Y}_j \cap B(\tilde{P}_j, |\tilde{P}_j \tilde{Q}_j|)$ converges to a vertical half-plane through \tilde{P}_j, \tilde{Q}_j , $|w_j|_{C^2} = o(1)$. Thus we obtain isothermal coordinates (q_j, w_j) on \tilde{Y}_j and a conformal factor e^{ϕ_j} which satisfies (4.4).

The final dilation and the jump in the derivative: Set $\tilde{Y}_j := B^\sharp \circ M_{t_j} \circ I(Y_j^{\text{ext}}) \cap \{x \geq 0\}$; this is a surface with boundary $\tilde{\gamma}_j = B^\sharp \circ M_{t_j}(Y_j') \cap \{x = 0\}$. Write $P_j := \tilde{\Psi}_j(0, 0)$ and $Q_j := \tilde{\Psi}_j(1, 0)$; these both lie on the line $\{y = 1, x = 0\}$, and since all maps here are conformal, $\tilde{\gamma}_j$ makes an angle bigger than $\alpha/2$ with this line at P_j . Translate and rotate so that P_j is the origin and $Q_j = (0, d_j, 0)$.

Let F_j be the Euclidean dilation from the origin by the factor d_j^{-1} , so that $F_j(Q_j) = (0, 1, 0)$. Note that $F_j(\tilde{\gamma}_j)$ still makes an angle bigger than $\alpha/2$ with the y -axis at $(0, 0, 0)$. Pre- and postcomposing $\tilde{\Psi}_j$ by F_j gives a conformal map

$$\hat{\Psi}_j := F_j \circ \tilde{\Psi}_j \circ F_j^{-1} : \mathbb{R}^2 \longrightarrow F_j(\tilde{Y}_j),$$

which leaves the conformal factor $\tilde{\phi}_j$ unchanged.

Since \tilde{Y}_j converge to a vertical plane, the curves $F_j(\tilde{\gamma}_j)$ converge in \mathcal{C}^α to the y -axis as $j \rightarrow \infty$. The required hyperbolic isometries φ_j are then just $\hat{\Psi}_j$, and the domains \mathcal{U}_j are the preimages of the graph of the unit disc under these maps.

Graphicality: We return finally to the claim that the surfaces \tilde{Y}_j remain graphical. Recall first that Y_j is graphical over the disc $\{x^2 + y^2 \leq 9, z = 0\}$, and that $|\nabla u_j| \leq 2\zeta$. Consider the family of straight lines $z \mapsto (x_0, y_0, z)$ parallel to the z -axis. Each of these meet \tilde{Y}_j at an angle $\Omega(x_0, y_0)$ which satisfies $|\Omega - \pi/2| < \arctan(2\zeta)$. Under the Möbius transformation I , this family is transformed to a family of circles \mathcal{C}_N , each passing through $(0, 0, 1)$ and meeting $S_1(0)$ orthogonally. By conformality, if C is one of these circles which intersects $I(\tilde{Y}_j)$ at a point Q with angle $\Phi(Q)$, then $|\Phi(Q) - \pi/2| \leq \arctan(2\zeta)$.

Now consider the set of circles \mathcal{C}_E passing through $(0, 1, 0)$ and intersecting $S_1(0)$ orthogonally. For any point $Q \in I(\mathcal{D}_j)$, consider the circles $C_N(Q) \in \mathcal{C}_N$ and $C_E(Q) \in \mathcal{C}_E$ which pass through Q . Since $\text{dist}(Q, S) \leq \arctan(1/10)$, the angle between $C_N(Q)$ and $C_E(Q)$, is at most $\arctan(2\zeta)$. Hence for every $Q \in I(\mathcal{D}_j)$, the angle $\Phi'(Q)$ between $C_E(Q)$ and the surface $I(\tilde{Y}_j)$ satisfies $|\Phi'(Q) - \pi/2| \leq 2\arctan(2\zeta)$.

Finally, note that the dilations M_t preserve the family \mathcal{C}_E . By conformality, for each point $Q \in M_{t_j}(I(\tilde{Y}_j))$, the surface $M_{t_j}(I(\tilde{Y}_j))$ makes an angle $\Phi'(Q)$ with $C_E(Q)$, where $|\Phi'(Q) - \pi/2| \leq 2\arctan(2\zeta)$. Recall that B^\sharp maps each $C_E(Q)$ to a line parallel to the z -axis. By conformality again, these lines make an angle $\Phi'(Q)$ with $B^\sharp \circ M_{t_j}(I(\tilde{Y}_j))$ at the point of intersection Q , where $|\Phi'(Q) - \pi/2| \leq 2\arctan(2\zeta)$. But this means precisely that $B^\sharp \circ M_{t_j}(I(\tilde{Y}_j))$ is a graph over a disk of some graph function \tilde{u}_j satisfying $|\nabla \tilde{u}_j| \leq 4\zeta$. \square

Construction of the extension.

We now prove the fact claimed in the proof of Lemma 4.1 and Remark ?? that the reflected surface Y_j' can be extended to a graph over the entire plane $\{z = 0\}$ in such a way that the increase of energy is controlled. This is straightforward using mollification. The point is that each of our surfaces is graphical with bounded tilt, so the total curvature is equivalent to the L^2 -norm of the Hessian of its graph function. In particular, if $Y = \text{Graph}(u)$ for $u \in \mathcal{C}^2(D')$, $D' = \{1/4 \leq x^2 + y^2 \leq 9\}$, with $|\nabla u| \leq 2\zeta$, then

$$\frac{1}{(1 + 4\zeta^2)} \int_{D'} |\partial^2 u|^2 dx dy \leq \int_{D'} |\bar{A}|^2 d\bar{\mu} \leq (1 + 4\zeta^2) \int_{D'} |\partial^2 u|^2 dx dy. \quad (4.11)$$

Lemma 4.2. *Let u be a $W^{2,2}$ function defined on the half-disc $D_+(3) := \{\sqrt{x^2 + y^2} \leq 3, x > 0\}$. If $Y = \text{Graph}(u)$ then write*

$$\int_Y |\bar{A}|^2 d\bar{\mu} := \mathcal{E}, \quad \int_{Y \cap \{1/2 \leq \sqrt{x^2 + y^2} \leq 3\}} |\bar{A}|^2 d\bar{\mu} := \mathcal{E}'$$

and assume that $|\nabla u| \leq 1$, and in addition

1. There exist $\epsilon, \delta > 0$ such that $|\nabla u(P)| \leq \delta$ for all $P \in D' \cap \{x \geq \epsilon\}$;
2. For any $P \in \overline{D'} \cap \{x = 0\}$, and any sequence $P_j \in D'$ with $P_j \rightarrow P$ we have $\lim_{j \rightarrow \infty} \partial_x u(P_j) = 0$.

Let U be the even extension of u to $D = \{\sqrt{x^2 + y^2} \leq 3\}$. Then there exists a function \bar{u} such that $\bar{u} = U$ on $\{\sqrt{x^2 + y^2} \leq 1\}$, $\bar{u} = 0$ on $\{\sqrt{x^2 + y^2} \geq 5\}$ and if we let $\bar{Y} := \text{Graph}(\bar{u})$ then $\int_{\bar{Y}} |\bar{A}|^2 d\bar{\mu} \leq 2\mathcal{E} + 1000(\delta + \epsilon) + 10\mathcal{E}'$.

By Remark 4.1 and the fact that the Y_j converge locally in \mathcal{C}^∞ to a vertical half-plane away from $\{x = 0\}$, this Lemma then implies the claim on extension from above.

Proof. First note that if $u \in W^{2,2}$, then using the fact that $\partial_x u = 0$ on $\{x = 0\}$, we have $U \in W^{2,2}$. Furthermore, using the formula for the second fundamental form of a graph $z = U(x, y)$, we have

$$\int_{Y'} |\bar{A}|^2 d\bar{\mu} = 2\mathcal{E}.$$

To construct the extension, fix a smooth cutoff function $\chi(x, y) \in \mathcal{C}_0^\infty(B_1(0))$ with $\int_{\mathbb{R}^2} \chi dx dy = 1$, and such that $|\partial \chi| \leq 10$, $|\partial^2 \chi| \leq 100$. Given any $\rho > 0$ we let $\chi_\rho := \rho^{-2} \chi(\frac{x}{\rho}, \frac{y}{\rho})$. We work in polar coordinates $r := \sqrt{x^2 + y^2}$, $\theta := \arctan(y/x)$.

Define a function $u^\sharp(r, \theta)$ which equals $u(r, \theta)$ for $r \leq 5/2$, and which vanishes for $r > 5/2$. In addition, let $\psi(r)$ be a \mathcal{C}^∞ function which vanishes when $r \leq 1$, equals 2 for $r \geq 3$, is strictly monotone increasing in the interval $[1, 3]$ and satisfies $|\psi'(r)| \leq 10$, $|\psi''(r)| \leq 100$. Then define the function

$$\bar{u}(r, \theta) := (u^\sharp * \chi_{\psi(r)})(r, \theta),$$

where χ_0 is understood as the δ function. It is straightforward that \bar{u} is \mathcal{C}^2 away from $\{x = 0\}$, and it is also obvious that $\bar{u}(r, \theta) = 0$ for $r \geq 5$ and that $|\nabla \bar{u}| \leq 1$ throughout \mathbb{R}^2 . What remains is to show that the surface $Y = \text{Graph}(\bar{u})$ satisfies the claims of our Lemma.

To do this, we recall some facts about the Hardy-Littlewood maximal functions. For $\chi \in \mathcal{C}_0^\infty$ and $f \in L_{loc}^1(\mathbb{R}^2)$, define

$$M(f)(x) := \sup_{\rho > 0} |(f * \chi_\rho)(x)|.$$

Then (for an appropriate choice of cutoff function χ),

$$\|M(f)\|_{L^2} \leq 10\|f\|_{L^2}. \quad (4.12)$$

Using (4.12) and (4.11) we derive:

$$\int_{Y \cap \{1 \leq r \leq 2\}} |\bar{A}|^2 d\bar{\mu} \leq 20\mathcal{E}'.$$

This implies immediately that

$$\int_{Y \cap \{r \leq 2\}} |\bar{A}|^2 d\bar{\mu} \leq 2\mathcal{E} + 20\mathcal{E}'.$$

Thus matters are reduced to estimating $\int_{Y_j \cap \{r \geq 2\}} |\bar{A}|^2 d\bar{\mu}$. Given (4.11), it suffices to control:

$$\int_{\{r \geq 2\}} |\partial^2 \bar{u}|^2 dx dy.$$

We use the formula $\partial^2 \bar{u}(P) = [\partial^2(\chi_{\psi(r(P))}) * u^\sharp](P)$. Using the pointwise bounds on $\partial^j \chi_r$, $\partial^j \psi$, $j = 0, 1, 2$ and on $|u|$, we directly derive that:

$$\int_{Y \cap \{2 \leq r \leq 5\}} |\bar{A}|^2 d\bar{\mu} \leq 1000 \int_{\mathbb{R}^2 \cap \{2 \leq r \leq 5\}} |\bar{u}|^2 dx dy \leq 1000(\epsilon + \delta)$$

Finally, using the definition of u^\sharp we deduce that $\int_{Y \cap \{5 \leq r\}} |\bar{A}|^2 d\bar{\mu} = 0$. \square

5 The key estimates: Small energy in a half-ball controls \mathcal{C}^1 regularity.

We now consider the setting of Proposition 3.1, and in particular consider the sequence of (incomplete, graphical) minimal surfaces furnished by 4.1. We perform a further dilation to ensure that these incomplete minimal surfaces are graphical over $D_+(3)$. As before, reset notation and denote these by Y_j .

We recall also the connection between the bounds on the conformal factor ϕ_j and on $|dq_j|_{\bar{g}}$ and $|dw_j|_{\bar{g}}$, as explained in Remark 4.3. As a consequence of this relationship, the bounds $\|\phi_j\|_{\mathcal{C}^0} \leq C_1$ and $\mathcal{E}[Y_j] \leq \epsilon'(\zeta) \ll 1$ imply that the image of the entire rectangle $0 \leq q_j \leq 1, |w_j| \leq 1$ lies in Y_j . Furthermore, since Y_j^b converges to a vertical plane Y_* (see Remark 4.2), and since q_j has gradient bounded above and below and vanishes along $\{x = 0\}$, and $\Delta_{\bar{g}_j} q_j = 0$, there is a subsequence of the q_j converging to a harmonic function of linear growth which vanishes at $x = 0$. The only possible limit is λx for some $\lambda \in [C^{-1}, C]$. As before, the convergence is \mathcal{C}^∞ away from $\{x = 0\}$.

Next, using the \mathcal{C}^0 and $W^{1,2}$ bounds for ϕ_j in (4.10), we may replace coordinate derivatives by covariant derivatives in these bounds, at worst only increasing the constant.

We are now in a position to prove the main analytic estimate, which shows that the energy of Y_j controls the jump in the first derivative of the boundary curve of Y_j at the origin.

In the following, we often write ∂_1 and ∂_2 for the coordinate vector fields ∂_{q_j} , ∂_{w_j} on Y_j . We also set $A_j(\partial_1, \partial_2) = (A_j)_{12}$, or simply A_{12} . Since the coordinate function z equals u_j on Y_j , we have

$$(\bar{A}_j)_{12} \nabla_\nu z = \nabla_{12} u_j = \partial_{12} u_j - \partial_1 u_j \partial_2 \phi_j - \partial_1 \phi_j \partial_2 u_j. \quad (5.1)$$

The two equalities are just specializations of the basic definitions to this situation. Now drop the subscript j for simplicity. Now multiply the equation by $e^{-\phi}$. Noting that $e^{-\phi}(\partial_{12} u - \partial_1 \phi \partial_2 u) = \partial_1(e^{-\phi} \partial_2 u)$, then integrating along the line segment $0 \leq q \leq 1, w = 0$ gives

$$(e^{-\phi} \partial_2 u)(1, 0) - (e^{-\phi} \partial_2 u)(0, 0) = \int_{(0,0)}^{(1,0)} e^{-\phi} \bar{A}_{12} \nabla_\nu u dq + \int_{(0,0)}^{(1,0)} e^{-\phi} \partial_2 \phi \partial_1 u dq. \quad (5.2)$$

We can now state our main estimate.

Proposition 5.1. *Set $\mathcal{E}_j := \mathcal{E}(Y_j)$. Then there exist constants C, C' such that*

$$\int_{(0,0)}^{(1,0)} |(\bar{A}_j)_{12} e^{-\phi_j}| dq_j + \int_{(0,0)}^{(1,0)} e^{-\phi_j} \partial_2 \phi_j \partial_1 z dq_j \leq C \sqrt{\mathcal{E}_j} + C' \mathcal{E}_j + o(1). \quad (5.3)$$

Before proving this Proposition, let us explain how they lead to a contradiction, thus establishing Proposition 3.1.

In view of the \mathcal{C}^∞ convergence of $u_j \rightarrow u_*$ and $(q_j, w_j) \rightarrow (q_*, w_*)$ (both for $x > 0$), we see that for j sufficiently large,

$$|e^{-\phi_j} \partial_{w_j} u_j(1, 0) - e^{-\phi_*} \partial_{w_*} u_*(1, 0)| \leq \alpha/10. \quad (5.4)$$

Furthermore,

$$e^{-\phi_j} \partial_{w_j} u_j(1, 0) - e^{-\phi_j} \partial_{w_j} u_j(0, 0) = \int_0^1 \partial_1(e^{-\phi_j} \partial_2 u_j) dq_j, \quad (5.5)$$

and since Y_* lies in the plane $\{z = 0\}$, we also have

$$e^{-\phi_*} \partial_{w_*} u_*(1, 0) - e^{-\phi_*} \partial_{w_*} u_*(0, 0) = 0. \quad (5.6)$$

Proposition 5.1 then yields that for large enough j

$$\int_{(0,0)}^{(1,0)} \partial_{q_j}(e^{-\phi_j} \partial_{w_j} u_j) dq_j \leq \alpha/10. \quad (5.7)$$

Combining these facts along with $|e^{-\phi_j} \partial_{w_j} u_j - \partial_y u_j| \leq \frac{1}{10} |\partial_y u_j|$, which holds since $|\nabla u_j| < 2\zeta \leq 1/10$ and $|\partial_{w_j} - \partial_y|_{\bar{g}}$ is also small, we conclude that

$$|\partial_y u_j(0, 0) - \partial_y u_*(0, 0)| \leq \alpha/3 \quad (5.8)$$

when j is large, which contradicts (4.2).

5.1 Regularity from the interior: the two line integrals

Proposition 5.1 is a consequence of the following two results:

Proposition 5.2. *With all notation as above, suppose that $\|\phi_j\|_{C^0(Y_j)} \leq K$ for all j . Then there exists a constant $C(K) > 0$ such that for each point $P = (x_0, y_0, u_j(x_0, y_0))$ in Y_j with $0 \leq x_0 \leq 1, |y_0| \leq 1/10$,*

$$|(\bar{A}_j)_{12}|(P) \leq C(K) \frac{\sqrt{\int_{Y_j} |A_j|^2 d\mu}}{\sqrt{|x(P)|}}. \quad (5.9)$$

Remark 5.1. *The proof of this proposition implies directly that $\int_{Y_j} |A_j|^2 d\mu$ may be replaced by $\int_{B_{\xi(P)}} |A_j|^2 d\mu_j$, where $\xi(P) = -\log x_0$.*

Proposition 5.3. *For some constants C, C' independent of j , we have*

$$\int_{(0,0)}^{(1,0)} e^{-\phi_j} \partial_2 \phi_j \partial_1 u_j dq_j \leq C \int_{Y_j} |\dot{\bar{A}}|^2 d\bar{\mu} + C' \sqrt{\int_{Y_j} |\dot{\bar{A}}|^2 d\bar{\mu}} + o(1); \quad (5.10)$$

5.2 Proof of Proposition 5.2: the integrable bound on A_{12}

We now turn to an examination of the decay properties of the off-diagonal component of the second fundamental form. Let us record immediately that since (q, w) is an isothermal chart, then there is a simple relationship $\bar{A}_{12} = xA_{12}$ between this component relative to the metrics induced from the Euclidean and hyperbolic ambient metrics. Thus it suffices to work with the second fundamental form relative to the complete metric g . There are general estimates for the pointwise behaviour of $|A|$ due to Choi-Schoen [6], but applying them in this setting we obtain the bound $|A|(P) \leq C$, or equivalently $|\bar{A}|(P) \leq C/x(P)$. Unfortunately the quantity on the right is not integrable, so more specifically adapted ideas must be used to obtain an integrable bound.

As before, suppress the index j . Recall that each Y is an incomplete minimal surface, with $\gamma := \partial_\infty Y$, and is a horizontal graphic over the vertical rectangle $\mathcal{R} = \{0 \leq x \leq 2, |y| \leq 2\}$. We know that $|\nabla u| \leq 4\zeta$, and in addition, a portion of Y is covered by an isothermal coordinate chart (q, w) , $\{0 \leq q \leq 1, |w| \leq 1\}$, and the corresponding conformal factor ϕ satisfies the bounds of Lemma 4.1. As before, we write $\partial_q = \partial_1$ and $\partial_w = \partial_2$, and consider $A_{12} = A(\partial_1, \partial_2)$.

Lemma 5.1. *This function A_{12} satisfies $\Delta_g A_{12} = 0$.*

Proof. We appeal to a classical result that if Y is a minimal surface in a space form, and if z is a local conformal coordinate chart, then $A(\partial_z, \partial_z)dz^2$ is a holomorphic quadratic differential. There are various proofs of this, see in particular the one by Chern [5], though he attributes the result to Calabi.

In our case, $z = q + iw$ and hence

$$4A(\partial_z, \partial_z) = A(\partial_1 - i\partial_2, \partial_1 - i\partial_2) = (A_{11} - A_{22}) - 2iA_{12},$$

is holomorphic, so the imaginary part is harmonic with respect to the coordinate z , and thence with respect to the metric g . \square

Lemma 5.2. $\Delta_g(x^4 A_{12}^2) \geq 0$.

Proof. We begin with the assertion that

$$\Delta_{\bar{g}}x = -x^{-1}(\nabla_{\bar{\nu}}x)^2 \leq 0, \quad (5.11)$$

where $\bar{\nu}$ is the Euclidean unit normal to Y .

To prove this, work in Fermi coordinates (x_1, x_2, t) around Y with respect to the ambient Euclidean metric \bar{g} ; here (x_1, x_2) is a coordinate system in Y and t is the signed distance to Y . In such coordinates, it is well known that

$$\Delta_{\text{euc}} = \partial_t^2 + H(\bar{g})\partial_t + \Delta_{\bar{g}(t)}, \quad (5.12)$$

where the last term on the right is the Laplacian with respect to the induced metric on the level set $\{t = \text{const.}\}$. At $t = 0$, this is simply the Laplacian $\Delta_{\bar{g}}$ on Y . Since $g = e^{2\phi}\bar{g}$ where $\phi = -\log x$, (1.7) asserts that $H(\bar{g}) = \partial_t x/x$.

Now, since x is linear, $\Delta_{\text{euc}}x = 0$. Using (5.12), we obtain that along Y ,

$$0 = \partial_t^2 x + H(\bar{g})\partial_t x + \Delta_{\bar{g}(t)}x.$$

The t coordinate curves are straight lines and x is linear, so $\partial_t^2 x = 0$, which gives that at $t = 0$,

$$\Delta_{\bar{g}}x = -H(\bar{g})\partial_t x = -x^{-1}(\partial_t x)^2 \leq 0,$$

as claimed.

Our next claim is that

$$\Delta_g x^2 \geq 0. \quad (5.13)$$

Since $\Delta_g = x^2 \Delta_{\bar{g}}$, it suffices to compute $\Delta_{\bar{g}}x^2$, and for this we use (5.11) and the same Fermi coordinate system to compute that

$$\Delta_{\bar{g}}x^2 = 2x\Delta_{\bar{g}}x + 2|\nabla x|_{\bar{g}}^2 = -2(\nabla_{\bar{\nu}}x)^2 + 2|\nabla x|_{\bar{g}}^2,$$

and this is nonnegative since the angle that the normal vector ∂_t to Y forms with ∂_z is at most $\arctan(2\zeta) < \pi/4$.

Finally, write $y = x^2$ and then compute:

$$\begin{aligned} \frac{1}{2}\Delta_g(y^2 A_{12}^2) &= (y\Delta_g y + |\nabla y|_g^2)A_{12}^2 + y^2|\nabla A_{12}|_g^2 + 2yA_{12}\nabla_g y \cdot \nabla_g A_{12} \\ &\geq |\nabla y|_g^2 A_{12}^2 + y^2|\nabla A_{12}|_g^2 + 2(A_{12}\nabla_g y) \cdot (y\nabla_g A_{12}) \geq 0, \end{aligned}$$

as claimed, where the Cauchy-Schwarz inequality gives the final inequality. \square

Recall that for a minimal surface $Y \subset \mathbb{H}^3$, the Codazzi equations imply that the Gauss curvature $R_Y \leq -1$. We now quote the well-known

Lemma 5.3 (Monotonicity Lemma). *Let (M^2, g) be a surface with sectional curvature $R_g \leq -1$. Then for any $P \in M$, and $f \in \mathcal{C}^2(B_P(R))$, where R is the injectivity radius at P , with $f \geq 0$ and $\Delta_g f \geq 0$, the function*

$$L(r) := \frac{\int_{B_P(r)} f \, d\mu}{\mu_{\mathbb{H}^2}(r)}$$

is nondecreasing for $0 < r \leq R$. Here $\mu_{\mathbb{H}^2}(r)$ is the area of the disc of radius r in the hyperbolic space \mathbb{H}^2 .

We now turn to the proof of Proposition 5.2. Fix $P = (x_0, y_0, u(x_0, y_0))$ as in the statement. Then the (hyperbolic) disc of radius $|\log x_0|$ is contained in the part of Y which is a graph over the box $0 \leq \sqrt{x^2 + y^2} \leq 2$. This follows from elementary hyperbolic geometry and the fact that the Euclidean projection of a hyperbolic disk of radius R in Y centered at P is contained in a hyperbolic disk of radius R in $\{(x, y, 0) : x > 0\}$.

Apply Lemma 5.3 based at the point $P = (x_0, y_0, u(x_0, y_0))$ and with $f = x^4 A_{12}^2$. The limit of the function $L(r)$ as $r \searrow 0$ is simply $x_0^4 A_{12}(P)^2$, hence setting $\rho = -\log x_0$ and recalling the area of discs in \mathbb{H}^2 , we have that for all points with $x_0 \leq 1/2$,

$$x_0^4 A_{12}(P)^2 \leq \frac{\int_{B_\rho(P)} x^4 A_{12}^2 \, d\mu}{\mu_{\mathbb{H}^2}(\rho)} \leq C x_0 \mathcal{E}(Y) \quad (5.14)$$

where $C = 4e^{2\|\phi\|_{C^0}}$. To see this observe that $|\partial_1|_{\bar{g}}, |\partial_2|_{\bar{g}} \leq e^{\|\phi\|_{C^0}}$, thus $x|\partial_1|_g, x|\partial_2|_g \leq e^{\|\phi\|_{C^0}}$. Setting $C = 4e^{2\|\phi\|_{C^0}}$, this implies that $x^4 A_{12}^2 \leq C|A|_g^2$.

Finally, using $(\bar{A})_{12} = x A_{12}$, we derive that

$$|(\bar{A})_{12}(P)| \leq C \frac{\sqrt{\mathcal{E}(Y)}}{\sqrt{x(P)}}.$$

When $1/2 \leq x_0 \leq 1$, the bound follows by the Choi-Schoen estimate [6]. \square

5.3 Proof of Proposition 5.3: the second line integral

We complete this analysis by establishing bounds for the second line integral in (5.3).

Since $e^{2\phi_j} = \bar{g}_j(\partial_1, \partial_1)$, we obtain that for j large enough, $|\phi_j| \leq 1/10$, and hence

$$e^{|\phi_j|} \leq 2, \quad \text{and} \quad 1/2 \leq |\partial_1|_{\bar{g}} \leq 2.$$

From these bounds we also obtain

$$|\nabla_1 u_j| \leq 4\zeta \implies -\frac{1}{2} \leq \partial_1 u_j \leq 2. \quad (5.15)$$

Recall also the basic equation, which follows from the Codazzi formulæ,

$$-\Delta_{\bar{g}} \phi = \frac{4\bar{H}^2 - |\bar{A}|^2}{4} = \frac{4\bar{H}^2 - |\bar{A}|^2}{8}, \quad (5.16)$$

as well as the identity

$$\Delta_{\bar{g}} e^{-\phi} = -\Delta_{\bar{g}} \phi e^{-\phi} + |\nabla \phi|_{\bar{g}}^2 e^{-\phi}. \quad (5.17)$$

The key for proving (5.10) is to express the integral I on the left in that formula as one of the boundary flux terms of an integration of the divergences of two vector fields over the two rectangles $\mathcal{D}_1 := \{0 \leq q \leq 1, 0 \leq w \leq 1\}$ and $\mathcal{D}_2 := \{0 \leq q \leq 1, -1 \leq w \leq 0\}$. To do this, introduce a cutoff function $\chi(w)$ such that $\chi \in \mathcal{C}^2$ with $0 \leq \chi \leq 1$, $\chi(0) = 1$, $\chi(-1) = \chi(1) = 0$, and such that $|\chi'| \leq 4$.

By Stokes' formula, and using the summation convention,

$$\begin{aligned}
I &= \int_{\mathcal{D}_1} \Delta_{\overline{g}} e^{-\phi} (\partial_1 u + 1) \chi d\overline{\mu} + \int_{\mathcal{D}_1} \partial_s e^{-\phi} \partial_s (\partial_1 u + 1) \chi dqdw \\
&+ \int_{\mathcal{D}_1} \partial_s (e^{-\phi}) (\partial_1 u + 1) \partial^s \chi d\overline{\mu} - \int \partial_1 e^{-\phi} (\partial_1 u + 1) \chi dw \Big|_{q=0}^{q=1} \\
&+ \int_{\mathcal{D}_2} \Delta_{\overline{g}} e^{-\phi} \chi d\overline{\mu} + \int_{\mathcal{D}_2} \partial_s e^{-\phi} e^{\phi} \partial_s \chi dqdw - \int \partial_1 e^{-\phi} e^{\phi} \chi dw \Big|_{q=0}^{q=1}
\end{aligned} \tag{5.18}$$

Some of these integrals are expressed with respect to the $d\overline{\mu}$ others with respect to the volume form $dqdw$, but the difference is not large since $d\overline{\mu} = e^{2\phi} dqdw$.

Since $Y_j \rightarrow Y_*$ smoothly away from $\{x = 0\}$, then for *any* $\eta \in (0, 1]$, $\partial_{q_j}|_{x=\eta} \rightarrow \partial_{q_*}|_{x=\eta}$ and $\phi_j|_{x=\eta} \rightarrow \phi_*|_{x=\eta}$ smoothly. The bounds on $|\partial_1|_{\overline{g}}$ and the fact that $\phi_* = \text{const}$ shows that

$$\int_{-1}^1 |\partial_1 \phi_j| dw_j \Big|_{q_j=1} = o(1). \tag{5.19}$$

On the other hand, by (4.4),

$$\int_0^1 \int_{-1}^1 |\partial_{12} \phi_j| dw_j dq_j \leq \mathcal{E}_j + o(1). \tag{5.20}$$

Combining these last two equations, we see that if $\eta \in (0, 1]$, then for $-1 \leq a \leq w \leq b \leq 1$,

$$\left| \int_a^b \partial_1 \phi_j dw_j \Big|_{q_j=\eta} \right| \leq \left| \int_a^b \partial_1 \phi_j dw_j \Big|_{q_j=1} \right| + \int_{\eta}^1 \int_a^b |\partial_{12} \phi_j| dw_j dq_j. \tag{5.21}$$

Since this is true for all subintervals $[a, b]$, then for each η we can divide the integral on the left into subintervals $[a, b]$ where $\partial_1 \phi_j|_{q_j=\eta}$ has constant sign, and then add these subintervals, to obtain that

$$\int_{-1}^1 |\partial_1 \phi_j| dw_j \Big|_{q_j=\eta} \leq \mathcal{E}_j + o(1), \tag{5.22}$$

where the error term is independent of η . Letting $\eta \rightarrow 0$ gives

$$\int_{-1}^1 |\partial_1 \phi_j| dw_j \Big|_{q_j=0} \leq \mathcal{E}_j + o(1). \tag{5.23}$$

Now consider the interior integral terms in (5.18). By (5.17), the first interior integral can be written as

$$\int_{\mathcal{D}_1} -\Delta_{\overline{g}_j} \phi_j (e^{-\phi_j}) (\partial_1 u_j + 1) \chi d\overline{\mu} + \int_{\mathcal{D}_1} |\nabla \phi_j|_{\overline{g}_j}^2 e^{-\phi_j} (\partial_1 u_j + 1) \chi d\overline{\mu}$$

The first term on the right here is controlled using (5.16):

$$\begin{aligned}
- \int_{\mathcal{D}_1} \Delta_{\overline{g}_j} \phi_j e^{-\phi_j} (\partial_1 u_j + 1) \chi d\overline{\mu} &= \int_{\mathcal{D}_1} \frac{|\mathring{A}_j|^2 - 4\overline{H}_j^2}{8} (\partial_1 u_j + 1) \chi e^{-\phi_j} d\overline{\mu} \\
&\leq 4 \int_{\mathcal{D}_1} \left(2|\mathring{A}_j|^2 - \frac{\overline{H}_j^2}{64} \right) \chi e^{-\phi_j} d\overline{\mu}.
\end{aligned} \tag{5.24}$$

Next, define

$$\mathcal{T} := \int_{\mathcal{D}_1} |\nabla \phi_j|_{\overline{g}_j}^2 e^{-\phi_j} (\partial_1 u_j + 1) \chi d\overline{\mu} + \int_{\mathcal{D}_1} \partial^s e^{-\phi_j} \partial_{s1} u_j \chi d\overline{\mu}.$$

Replace $\partial_{1s}u_j$ by $\nabla_{1s}u_j$ using $\nabla_{ab}u_j = \partial_{ab}u_j - \Gamma_{ab}^t \partial_t u_j$, where

$$\Gamma_{21}^1 = \partial_1 \phi_j, \Gamma_{22}^2 = \partial_2 \phi_j, \Gamma_{21}^2 = -\partial_2 \phi_j, \Gamma_{22}^1 = \partial_1 \phi_j, \Gamma_{12}^1 = \partial_2 \phi_j, \Gamma_{12}^2 = \partial_1 \phi_j, \quad (5.25)$$

to get that \mathcal{T} equals

$$\begin{aligned} & \int_{\mathcal{D}_1} |\nabla \phi_j|_{\bar{g}_j}^2 e^{-\phi_j} (\nabla_1 u_j + 1) \chi d\bar{\mu} + \int_{\mathcal{D}_1} \partial^s e^{-\phi_j} \nabla_{s1} u_j \chi d\bar{\mu} \\ & + \int_{\mathcal{D}_1} \partial^s e^{-\phi_j} \Gamma_{s1}^t \partial_t u_j \chi d\bar{\mu} = \int_{\mathcal{D}_1} \left(\partial^s e^{-\phi_j} \nabla_{s1} u_j \chi |\nabla \phi_j|_{\bar{g}_j}^2 e^{-\phi_j} \chi \right) d\bar{\mu}. \end{aligned} \quad (5.26)$$

Applying Cauchy-Schwarz, (4.4) and $|\nabla_{ab}u_j|_{\bar{g}_j} \leq 2|(\bar{A}_j)_{ab}|_{\bar{g}_j}$ we derive:

$$\mathcal{T} \leq 100 \int_{\mathcal{D}_1 \cup \mathcal{D}_2} |\nabla \phi_j|^2 \chi d\bar{\mu} + \frac{1}{100} \int_{\mathcal{D}} |\bar{A}_j|^2 \chi d\bar{\mu} \leq C200 \mathcal{E}_j + \frac{1}{50} \int_{\mathcal{D}} |\bar{H}_j|^2 d\bar{\mu} + o(1). \quad (5.27)$$

As for the third bulk term, using (4.4) again, we derive

$$\begin{aligned} \mathcal{Z} &:= \int_{\mathcal{D}_1} e^{-2\phi_j} \partial_2 e^{-\phi_j} (\partial_1 u_j + 1) \partial_2 \chi d\bar{\mu} = \int_{\mathcal{D}_1} \partial_2 e^{-\phi_j} (\partial_1 u_j + 1) \partial_2 \chi dqdw \\ &\leq 4 \sqrt{2 \int_{\mathcal{D}_1} |\nabla \phi_j|_{\bar{g}_j}^2 d\bar{\mu}} \cdot \sqrt{\int_{\mathcal{D}_1} dqdw} \leq 10C \sqrt{\mathcal{E}_j} + o(1). \end{aligned} \quad (5.28)$$

We control the last two bulk terms by

$$\int_{\mathcal{D}_2} (-\Delta_{\bar{g}_j} \phi e^{-\phi_j} + |\nabla \phi_j|_{\bar{g}_j}^2 e^{-\phi_j}) \chi d\bar{\mu} \leq \int_{\mathcal{D}_2} \frac{|\bar{A}_j|^2 - \bar{H}_j^2}{4} |\chi| d\bar{\mu} + 2C\mathcal{E}_j + o(1). \quad (5.29)$$

Finally, the Cauchy-Schwarz inequality together with (4.4) one last time gives

$$\int_{\mathcal{D}_2} e_j^{-2\phi} \partial_2 \phi_j \partial_2 \chi d\bar{\mu} \leq 2 \sqrt{\int_{\mathcal{D}} |\nabla e^{-\phi_j}|^2 d\bar{\mu}} \sqrt{\int_{\mathcal{D}_2} dqdw} \leq 2 \sqrt{\int_{\mathcal{D}_2} |\bar{A}_j|^2 d\bar{\mu}}. \quad (5.30)$$

Taken together, these estimates complete the proof. The only thing to observe is that the terms $\int_{\mathcal{D}} \bar{H}_j^2 d\bar{\mu}$ appears with a *negative* coefficient in the end, and so can be discarded, since our proposition only claims an upper bound on I . \square

6 Regularity gain for the limit surface in the small energy regions

We now turn to a closer look at the relationship between finiteness of the energy and the regularity of the boundary curve at infinity, and prove Theorem 1.4. In fact we prove the C^1 regularity for all minimal surfaces of finite total curvature near points where the boundary curve is locally graphical and Lipschitz. By this we mean the following:

Definition 6.1. *Consider a rectifiable closed embedded loop $\gamma \subset \mathbb{R}^2$, given in arc-length parametrization as $t \rightarrow (y(t), z(t)) = \gamma(t)$. We say that γ is locally Lipschitz at a point $P = \gamma(t_0)$ if there exists a $\delta(t_0) > 0$ and a constant $M(t_0) < \infty$ such that, after a rotation, $\gamma|_{t \in (t_0 - \delta, t_0 + \delta)}$ can be expressed as the graph of a function $z = f(y)$ over an interval of length $\eta(t_0)$ centered at $\gamma(t_0)$, so $\gamma|_{t \in (t_0 - \delta, t_0 + \delta)} = \text{Graph}(f)$ and $|f(y_1) - f(y_2)| \leq M(t_0)|y_1 - y_2|$.*

Our main result in this section is the

Theorem 6.1. *Let $Y \subset \mathbb{H}^3$ be a complete minimal surface with $\gamma = \partial_\infty Y$ a possibly disconnected embedded rectifiable curve. Suppose that $\mathcal{E}(Y) < \infty$ and that γ is locally graphical and Lipschitz except at a finite number of points $\{P_1, \dots, P_N\}$. Assume finally that if $\gamma(t_0) \notin \{P_1, \dots, P_N\}$, $M(t_0) = \zeta$ and $\mathcal{E}^{B(\gamma(t_0), \delta(t_0))}(Y) \leq \epsilon'(\zeta)$. Then $\gamma \setminus \{P_1, \dots, P_N\}$ is a \mathcal{C}^1 curve.*

Note that this subsumes the setting in Theorem 1.4 for the curve $\partial_\infty Y_*$ which arises as the limit of the $\partial_\infty Y_j$, since Corollary 1.1 ensures graphicality and a Lipschitz bound for the limiting curve away from a finite number of bad points P_1, \dots, P_N .

Theorem 6.1 is a consequence of the

Proposition 6.1. *Let $\gamma_k(t)$, $0 < t < M_k$, be an arclength parametrization of the k^{th} connected component of γ . Choose any Cauchy sequence $t_j \in (0, M_k)$ with $t_j \rightarrow t_* \in (0, M_k)$. Then $\dot{\gamma}(t_j)$ is a Cauchy sequence.*

Proof. Since $\gamma(t_*)$ is not equal to one of the P_j , there exists a line ℓ through $\gamma_k(t_*)$ and a number δ such that $\gamma|_{(t_*-\delta, t_*+\delta)}$ is a graph over the interval of length δ in ℓ centered at $\gamma(t_*)$ with graph function $z = f(y)$ having Lipschitz constant ζ . Lemma 2.3 guarantees graphicality of $Y'_{B(\gamma(t_0), h\delta)}$ over the region $\sqrt{x^2 + y^2} \leq h\delta$ in the vertical half-plane $\ell \times \mathbb{R}^+$ (where we take ℓ as the y -axis), with graph function $z = u(x, y)$, where $|\nabla u| \leq 2\zeta$.

Since t_j is Cauchy, it lies in $(t_* - h\delta, t_* + h\delta)$ for j large, so if we write $\gamma(t_j) = (y_j, u(0, y_j))$, then $y_j \rightarrow 0$.

Now, argue by contradiction and assume that $\dot{\gamma}(t_j)$ is not Cauchy. Then there exists $\theta > 0$ and a subsequence j_k such that $|f'(y_{j_{k-1}}) - f'(y_{j_k})| \geq \theta$. Reset notation and let the index be simply j again. Translate and rotate so that $(y_{j-1}, f(y_{j-1})) = (0, 0)$ and $(y_j, f(y_j))$ lies on the y -axis, then dilate by the factor $\lambda_j := |y_j - y_{j-1}|^{-1}$. Denote the resulting minimal surface by \tilde{Y}_j and write $\partial \tilde{Y}_j = \tilde{\gamma}_j$.

This surface is still graphical with Lipschitz norm no larger than ζ , and furthermore, $\mathcal{E}^{B(0, \lambda_j h\delta)}(\tilde{Y}_j) \leq \epsilon'(\zeta)$. By Lemma 2.2, \tilde{Y}_j must converge to a vertical half-plane Y_* , and since $\partial_\infty Y_*$ passes through the origin and $(0, 1, 0)$, necessarily $Y_* = \{z = 0\}$. Thus $Y_* \cap \{x = 1\}$ must converge to the line $\{z = 0, x = 1\}$ for some α with $|\alpha| \leq 2\zeta$. But now, since $|f'_j(0) - f'_j(1)| \geq \theta$, it follows that for at least one of the two values $y = 0, y = 1$ there is a jump in the derivative of size at least $\theta/2$ between the heights $x = 0$ and 1 . We can assume that this jump occurs at $y = 0$.

However, this contradicts Proposition 3.2. The graphicality and Lipschitz bound in that Proposition still hold by virtue of the assumption and Lemma 2.3. The fact that the energy goes to zero follows from the dilation invariance of \mathcal{E} , and the fact that after dilation, the graphs satisfy $\mathcal{E}(\text{Graph}(u_j)) \leq \mathcal{E}(Y \cap B(P, 2h\lambda_j^{-1}))$, and $\lambda_j^{-1} = |y_j - y_{j-1}| \rightarrow 0$. This proves the Proposition and Theorem 6.1 as well. \square

7 Bubbling in the small energy regions.

We now turn to a closer examination of how bubbling occurs. The main goal here is to prove Theorem 1.5.

As explained in the introduction, this proof of bubbling is indirect. We start by constructing a sequence of Möbius transformations φ_j to obtain the uniform isothermal parametrization. If these blown-up surfaces $\varphi_j(Y_j)$ converge to a non-trivial surface, we are done. Otherwise, we must prove that a nontrivial limit can be obtained through some further sequence of dilations.

The idea is to use the jump in the first derivative coupled with the bounds (7.4) to argue that one of the two line integrals on the right side of that inequality must be bounded below. If the rescaled surfaces $\varphi_j(\tilde{Y}_j)$ were to converge to a totally geodesic surface, then the arguments in the initial subsection of §5 treating the flux formula show that the second line integral must have this property. Unlike

in the final subsection of §5, it is not enough to bound this line integral by the energy in a half-ball. Fortunately, we can bound it in terms of the energy in a sector $|\frac{w}{q}| \leq 1$ emanating from the distinguished boundary point in the isothermal coordinates (q, w) . This bound can then be used to show the existence of a sequence of points where either $|x\bar{A}|$ or $|\nabla\phi_j|_g$ are bounded away from zero. Either alternative provides the points around which we can recenter the rescalings. In the first case, we obtain a limit surface with non-zero curvature at one interior point, which must therefore be nontrivial. In the second we obtain a complete minimal surface for which the canonical isothermal coordinates have non-constant conformal factor, and which are therefore also nontrivial. The key difficulty in bounding the left side of (7.5) in terms of the energy in a sector is that the cutoff function depends on w/q , so a derivative of this cutoff function produces an power of $1/x$. The resulting integral is controlled by using the specific algebraic form of the integrand on the left in (7.5). This somewhat remarkable fact is further evidence of the delicate nature of the blow-up procedure.

Proof. First, by translating and dilating, assume that $y_0 = 0$, and that the Y_j and Y_* are graphical over the vertical half-disc $\{x^2 + y^2 \leq 1000, z = 0\}$, with graph functions u_j and u_* , where $|\nabla u_j|, |\nabla u_*| \leq 2\zeta \ll 1$. We can also assume that $\partial_y u_j(0, 0) = \alpha > 0$ and $\partial_y u_*(0, 0) = 0$. Now, consider any subinterval $[\beta_-, \beta_+] \subset [0, \alpha]$ with $\beta_+ \ll \alpha$. For any sequence $\beta_j \in [\beta_-, \beta_+]$, $\beta_j \rightarrow \beta_*$, the line $z = \beta_j y$ intersects $\gamma_j = \partial_\infty Y_j$ at a point $(y_j, u_j(0, y_j))$, $y_j > 0$, and just as in §4, we have $\lim_{j \rightarrow \infty} y_j = 0$.

Now dilate Y_j by $\rho_j := \frac{1}{y_j}$ to obtain a new surface \tilde{Y}_j which converges to Y' , where Y' is graphical over the entire vertical half-plane $\{z = 0\}$ and passes through the fixed point $(1, \beta_*)$. If, for *any* such sequence β_j , $\mathcal{E}(Y') \neq 0$, then the proof is complete.

Otherwise, $\mathcal{E}(Y') = 0$ so Y' is totally geodesic and graphical over a half-plane, hence is the half-plane $\{z = \beta_* y, x > 0\}$. Rotating again to make this the xy -plane, the original graph function u_j must satisfy $\partial_y u_j(0, 0) = 0$ while $\partial_y u_*(0, 0) \sim \alpha - \beta_* > \frac{\alpha}{2} > 0$. All of this is true for any $\beta_* \in [\beta_-, \beta_+]$. Using Remark 4.1, there exists a sequence β_j such that $\int_{\tilde{Y}_j \cap \{1/4 \leq \sqrt{x^2 + y^2 + z^2} \leq 4\}} |A^j|^2 d\bar{\mu} \rightarrow 0$.

By Lemma 4.1, there exists a sequence of hyperbolic isometries φ_j such that $\varphi_j(\tilde{Y}_j)$ have all the properties listed there, and in particular admit isothermal coordinates (q_j, w_j) for which the conformal factor ϕ_j satisfies

$$\begin{aligned} \|\nabla^2 \phi_j\|_{L^1(\varphi_n(\tilde{Y}_j))} + \|\nabla \phi_j\|_{L^2(\varphi_j(\tilde{Y}_j))} + \|\phi_j\|_{C^0(\varphi_j(\tilde{Y}_j))} \\ \leq \mathcal{E}(\varphi_j(\tilde{Y}_j)) + o(1) < 2\epsilon'(\zeta) + o(1). \end{aligned} \quad (7.1)$$

Moreover, there is still a jump of $\alpha - \beta_*$ in the first derivative at the origin in these coordinates. The $\varphi_j(\tilde{Y}_j)$ are graphical over the disc $\{x^2 + y^2 \leq 10, z = 0\}$ (for simplicity, we denote the graph function by u_j) with $|\nabla u_j| \leq 4\zeta$ and the image of the rectangle $0 \leq q_j \leq 1, |w_j| \leq 1$ is entirely contained in $\text{Graph}(u_j)$. Recall from Remark 4.2 that the surfaces $\varphi_j(\tilde{Y}_j)$ admit an extension Y_j^b which is a graph over the xy -plane with graph function u_j , where $u_j = 0$ for $\sqrt{x^2 + y^2} \geq 50$. The bounds (7.1) continue to hold for this extended surface.

Using Remark 4.3 and the smallness of the energy we derive that $|y|/x \leq 10$ and $x^2 + y^2 \leq 10$ at all points in the sector

$$\mathcal{S}_j := \{(q_j, w_j) \in Y_j, |w_j|/q_j \leq 1, q_j^2 + w_j^2 \leq 4\},$$

We now claim that one of the following must be true:

- a) Either $\varphi_n(\tilde{Y}_j)$ converge to a *nontrivial* limit \tilde{Y}_* , or else
- b) there exists a sequence $\omega_j \rightarrow \infty$ such that the dilates $\omega_j \cdot \varphi_n(\tilde{Y}_j)$ converge to a non-trivial limit \tilde{Y}_* .

The theorem will be proved once we show that these are the only possibilities.

As many times before, write $\varphi_j(\tilde{Y}_j)$ as just Y_j . If alternative a) does not occur, then Y_j converges to a vertical half-plane Y_* . We claim that for some $\mu > 0$, there exists a sequence $P_j \in \mathcal{S}_j$ such that either $|\nabla\phi_j|_g(P_j) \geq \mu$ or else $x|\bar{A}|_{\bar{g}}(P_j) \geq \mu$.

Suppose the former of these is true and consider the dilated surfaces $\frac{1}{x(P_j)}Y_j$. The images \tilde{P}_j of the points P_j have height $x_j = 1$ and $|y_j| \leq 10$. Setting $\lambda_j := \frac{1}{x(P_j)}$, consider the isothermal coordinates

$$\tilde{q}_j(\lambda_j x, \lambda_j y) = \lambda_j q_j(x, y), \quad \tilde{w}_j(\lambda_j x, \lambda_j y) = \lambda_j w_j(x, y),$$

and the corresponding conformal factor $\tilde{\phi}_j(\lambda_j x, \lambda_j y) = \phi_j(x, y)$ on $\lambda_j Y_j$. Clearly, $|\nabla\tilde{\phi}_j|_{\tilde{g}}(\tilde{P}_j) \geq \mu$. Also, passing to a subsequence, $\tilde{P}_j \rightarrow \tilde{P}_*$ where $x(\tilde{P}_*) = 1$, $|y(\tilde{P}_*)| \leq 10$ and $|z(\tilde{P}_*)| \leq 10\zeta$.

By [6], some subsequence of the surfaces $\lambda_j Y_j$ must converge, smoothly in the interior and in $\mathcal{C}^{0,\alpha}$ up to the boundary, to a surface \tilde{Y}_* , and this limit surface admits isothermal coordinates $(\tilde{q}_*, \tilde{w}_*)$ where $\tilde{q}_* = 0$ on $\{x = 0\}$ and $1/10 \leq |\tilde{q}|/|x| \leq 10$. The convergence $(\tilde{q}_j, \tilde{w}_j, \tilde{\phi}_j) \rightarrow (\tilde{q}_*, \tilde{w}_*, \tilde{\phi})$ is smooth away from $\{x = 0\}$. We claim that \tilde{Y}_* can not be a vertical half-plane. Indeed, if it were, then following the same argument as in the second paragraph of §5, $\tilde{q}_* = Cx$ for some $1/10 \leq C \leq 10$, and in that case, the corresponding conformal factor $\tilde{\phi}_*$ would be constant. This contradicts the smooth convergence and the fact that $|\nabla\tilde{\phi}_j(\tilde{P}_j)|_{\tilde{g}} \geq \mu$.

The proof that $x|\bar{A}|_{\bar{g}}(P_j) \geq \mu$ implies the result is even simpler. Indeed, the same sequence of dilations of Y_j converges to a minimal surface with $|\bar{A}|_{\bar{g}} \geq \mu$, and this must be nontrivial since we know that it is graphical over the half-plane $\{z = 0\}$ and hence cannot be a sphere.

We have therefore reduced the proof to showing that conditions i) - vi) below lead to a contradiction.

- i) Each Y_j is a graphical minimal surface, with graph function u_j , over $\{x^2 + y^2 \leq 10, x > 0, z = 0\}$, with $\mathcal{E}(Y_j) \leq \epsilon'(\zeta)$ and $\int_{Y_j} |\bar{A}_j|^2 d\bar{\mu} \leq M$ for some fixed $M < \infty$. The surface Y_j extends to a (nonminimal) graphical surface Y_j^b . The region $30 \leq \sqrt{x^2 + y^2}$ is denoted Y_j^\sharp and $u_j = 0$ there.
- ii) Each Y_j , and its extension Y_j^b too, admits an isothermal coordinate chart (q_j, w_j) with conformal factor ϕ_j satisfying (7.1).
- iii) $Y_j \rightarrow Y_* := \{x^2 + y^2 \leq 10, z = 0\}$.
- iv) $\partial_y u_j(0, 0) - \partial_y u_*(0, 0) \geq \frac{\alpha}{2} > 0$.
- v) The conformal factors ϕ_j satisfy $|\nabla\phi_j|_g \rightarrow 0$ uniformly in \mathcal{S}_j .
- vi) $x \cdot |\bar{A}_j|_{\bar{g}} \rightarrow 0$ uniformly in \mathcal{S}_j .

The rectangle $0 \leq q_j \leq 1, |w_j| \leq 1$ lies in Y_j , and the monotonicity formula in Lemma 5.3 shows that for each point P in this rectangle

$$|(\bar{A}_j)_{12}|(P) \leq C\sqrt{\mathcal{E}(B'(P))}/\sqrt{x(P)}, \quad (7.2)$$

where $B'(P)$ is the intersection of Y_j with the hyperbolic ball of radius $-\log 2x(P)$. Note that $\mathcal{E}(B'(P)) \leq \epsilon'(\zeta)$, and if $\xi > 0$, then for all P with $x(P) \geq \xi$, $\mathcal{E}(B'(P)) \rightarrow 0$, which holds since Y_j converges to a hemisphere or vertical half-plane smoothly away from $\{x = 0\}$. Hence for any sequence of curves $\sigma_j \subset Y_j$ which are \mathcal{C}^1 with respect to (q_j, w_j) and which connect $(0, 0)$ to some point in the sector \mathcal{S}_j with $q_j = 1$, with \mathcal{C}^1 norm and length bounded above, we have

$$\int_{\sigma_j} |(\bar{A}_j)_{12}|_{\bar{g}} \rightarrow 0. \quad (7.3)$$

Now recall that when j is large,

$$e^{-\phi_j} \partial_2 u_j|_{(q_j, w_j)=(1,0)} - e^{-\phi_j} \partial_2 u_j|_{(q_j, w_j)=(0,0)} \sim \alpha - \beta_*,$$

and in addition,

$$\begin{aligned} e^{-\phi_j} \partial_2 u_j \Big|_{(q_j, w_j)=(1,0)} - e^{-\phi_j} \partial_2 u_j \Big|_{(q_j, w_j)=(0,0)} \\ \leq \int_{(0,0)}^{(1,0)} e^{-\phi_j} |(\bar{A}_j)_{12}| dq_j + \int_{(0,0)}^{(1,0)} \partial_2 e^{-\phi_j} \partial_1 u_j dq_j. \end{aligned} \quad (7.4)$$

As a special case of (7.3), we have

$$\int_{(0,0)}^{(1,0)} e^{-\phi_j} |(\bar{A}_j)_{12}| dq \rightarrow 0,$$

so to reach the contradiction it suffices to prove that under conditions i) - vi),

$$\lim_{j \rightarrow \infty} \left| \int_{(0,0)}^{(1,0)} \partial_2 e^{-\phi_j} \partial_1 u_j dq_j \right| = 0. \quad (7.5)$$

Proof of (7.5): Recall that since the conformal factor ϕ_j is bounded, the quantities $|\partial_q|$ and $|\partial_x|$, and $|\partial_y|$, $|\partial_w|$ are comparable. In the following, $|\cdot|$ denotes the norm with respect to $dq^2 + dw^2$. In many expressions below, we suppress the subscripts j for simplicity.

The strategy is to express the second integral $\int_{(0,0)}^{(1,0)} \partial_2 e^{-\phi_j} \partial_1 u dq$ as the flux of the integral of a divergence over some part of the circular sector \mathcal{S}_j . Introduce polar coordinates $r_j = \sqrt{q_j^2 + w_j^2}$ and θ_j with $\tan(\theta_j + \frac{\pi}{2}) = \frac{w_j}{q_j}$, so that $\mathcal{S}_j := \{0 \leq r_j \leq 1, \pi/4 \leq \theta_j \leq 3\pi/4\} \subset Y_j$. Let \mathcal{S}^l denote the region where $\pi/2 \leq \theta \leq 3\pi/4$, and define $\chi^l(\theta) = (3 - \frac{4\theta}{\pi})$ in \mathcal{S}^l .

By the divergence theorem,

$$\begin{aligned} \int_{(0,0)}^{(1,0)} \partial_2 e^{-\phi_j} \partial_1 u dq &= \int_{\mathcal{S}^l} \Delta_{\bar{g}} e^{-\phi_j} \partial_1 u \chi^l d\bar{\mu} + \int_{\mathcal{S}^l} \nabla_s e^{-\phi_j} \nabla^s (\partial_1 u) \chi^l d\bar{\mu} \\ &+ \frac{4}{\pi} \int_{\mathcal{S}^l} e^{-2\phi_j} \partial_\theta e^{-\phi_j} \frac{1}{r^2} \partial_1 u d\bar{\mu} + \int_{\pi/2}^{3\pi/4} (\partial_1 u) \partial_1 e^{-\phi_j} (1, \theta) d\theta. \end{aligned} \quad (7.6)$$

(The coefficient $\frac{4}{\pi}$ arises from $\partial_\theta \chi^l$.) The final boundary term tends to zero since Y_j converges to a vertical half-plane, so in particular $|\partial u_j|, |\partial \phi_j| \rightarrow 0$ away from $\{x = 0\}$.

Now consider the bulk terms. First, observe that the pointwise bounds on ϕ_j and on $|\nabla u|_{\bar{g}}$ imply that $|\partial_1 u|_{\bar{g}} \leq 3\zeta$. This uses the formula for the second fundamental form for a graph in \mathbb{R}^3 and implies that:

$$\int_{\mathcal{S}^l} |\partial^2 u|^2 d\bar{\mu} \leq 10 \int_{\mathcal{S}^l} |\bar{A}|^2 d\mu \leq 10M. \quad (7.7)$$

Using (5.16) and $|\partial_r u|_{\bar{g}} \leq 3\zeta$, we have

$$\begin{aligned} \left| \int_{\mathcal{S}^l} \Delta_{\bar{g}} e^{-\phi} \partial_1 u \chi^l d\bar{\mu} \right| \\ \leq 4 \int_{\mathcal{S}^l} e^{-\phi} \bar{H}^2 |\partial_1 u| d\bar{\mu} + \int_{\mathcal{S}^l} |\bar{A}|^2 e^{-\phi} |\partial_1 u| d\bar{\mu} + \int_{\mathcal{S}^l} |\nabla \phi|^2 e^{-\phi} d\bar{\mu}. \end{aligned} \quad (7.8)$$

Clearly,

$$\int_{\mathcal{S}^l} \bar{H}^2 |\partial_1 u| d\bar{\mu} + 4 \int_{\mathcal{S}^l} |\bar{A}|^2 e^{-\phi} |\partial_1 u| d\bar{\mu} \leq 10 \int_{\mathcal{S}^l} |\bar{A}|^2 |\partial_1 u| d\bar{\mu}. \quad (7.9)$$

In addition, using (7.1) and (7.7),

$$\begin{aligned} \left| \int_{S^l} \nabla_s e^{-\phi} \nabla^s (\partial_1 u) \chi^l d\bar{\mu} \right| &= \left| \int_{S^l} e^{-2\phi} \partial_s e^{-\phi} \partial^s (\partial_1 u) \chi^l d\bar{\mu} \right| \\ &\leq 4 \left(\int_{S^l} |\nabla \phi|^2 d\bar{\mu} \right)^{\frac{1}{2}} \left(\int_{S^l} |\partial^2 u|^2 d\bar{\mu} \right)^{\frac{1}{2}} \leq 100\sqrt{M} \left(\int_{S^l} |\nabla \phi|^2 d\bar{\mu} \right)^{\frac{1}{2}}. \end{aligned} \quad (7.10)$$

The main issue is to control the term

$$T_2 := \left| \int_{S^l} \frac{1}{r^2} e^{-2\phi} \partial_\theta e^{-\phi} \partial_1 u d\bar{\mu} \right|.$$

Recall that by Lemma 2.1, $\partial_1 u = 0$ on $\{q = 0\} = \{x = 0\}$, and also $\frac{|\partial_1 u|^2}{r^2} \leq \frac{|\partial_1 u|^2}{q^2}$. The Hardy inequality now gives

$$\int_{S^l} \frac{|\partial_1 u|^2}{r^2} dq dw \leq 10 \int_{\mathcal{B}} \frac{|\partial_1 u|^2}{q^2} dq dw \leq \int_D |\partial^2 u|^2 dq dw \leq 100 \int_{Y_j} |\bar{A}|_{\frac{2}{g}}^2 d\bar{\mu} \leq 100M. \quad (7.11)$$

where $\mathcal{B} := \{0 \leq w \leq 1, 0 \leq q \leq 1\}$. Thus

$$\begin{aligned} T_2 &\leq 10 \int_{S^l} r^{-2} |\partial_\theta \phi \partial_1 u| d\bar{\mu} \leq 10 \left(\int_{S^l} |\nabla \phi|_{\frac{2}{g}}^2 d\bar{\mu} \right)^{\frac{1}{2}} \left(\int_{S^l} r^{-2} |\partial_1 u|^2 d\bar{\mu} \right)^{\frac{1}{2}} \\ &\leq 100 \left(\int_{S^l} |\nabla \phi|_{\frac{2}{g}}^2 d\bar{\mu} \right)^{\frac{1}{2}} \left(\int_{\mathcal{B}} |\partial^2 u|^2 d\bar{\mu} \right)^{\frac{1}{2}} \leq 100\sqrt{M} \left(\int_{S^l} |\nabla \phi|_{\frac{2}{g}}^2 d\bar{\mu} \right)^{\frac{1}{2}}. \end{aligned} \quad (7.12)$$

We then claim that

$$\lim_{j \rightarrow \infty} \int_{S^l} |\nabla \phi_j|_{\frac{2}{g}}^2 d\bar{\mu} = 0, \quad (7.13)$$

and

$$\lim_{j \rightarrow \infty} \int_{S^l} |\bar{A}_j|^2 |\partial_1 u_j| d\bar{\mu} = 0. \quad (7.14)$$

These estimates will prove (7.5), and thus our theorem.

Proof of (7.13): We assert first that on the family of lines $\ell_{\theta_0} := \{0 \leq r \leq 1, \theta = \theta_0\}$, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$, there is a uniform bound

$$\int_{\ell_{\theta_0}} |\partial \phi_j| dr \leq \epsilon'(\zeta) + M'. \quad (7.15)$$

Before proving this, let us see how this proves the estimate.

Since $1/10 \leq q/x \leq 10$ in \mathcal{S}^l , we have

$$r |\nabla \phi_j|_{\frac{2}{g}} \leq 100 |\nabla \phi_j|_g$$

in this sector, so that

$$\begin{aligned} \int_{S^l} |\nabla \phi_j|_{\frac{2}{g}}^2 d\bar{\mu} &= \int_{\pi/2}^{3\pi/4} \int_0^1 |\partial \phi_j| |\partial \phi_j| r dr d\theta \\ &\leq 100 \sup_{S^l} |\nabla \phi_j|_g \sup_{\theta \in [\pi/2, 3\pi/4]} \int_{\ell_\theta} |\partial \phi_j| dr. \end{aligned}$$

Since the first factor tends to 0 by the assumption v) above and the second one is bounded, we obtain (7.13).

Thus matters are reduced to showing (7.15). Recall from (7.1) that $\|\partial^2 \phi_j\|_{L^1(\mathbb{R}^2)} \leq \epsilon'(\zeta) + o(1)$, where ∂ is differentiation with respect to (q_j, w_j) . Given any ray ℓ_{θ_0} ,

consider the right-angle rectangle $R_{\theta_0} \subset Y_j^\flat$ which is defined by four straight (with respect to the coordinates q_j, w_j) line segments: ℓ_{θ_0} is one segment, then s^1, s^2 are two line segments of length 50, emanating from the endpoints $(0, 0)$ and $(\cos \theta_0, \sin \theta_0)$ of ℓ_{θ_0} and normal to it; finally ℓ'_{θ_0} joins the other endpoints of s^1, s^2 . Thus ℓ'_{θ_0} is parallel to ℓ_{θ_0} (with respect to the flat coordinates q_j, w_j) and lies in the portion Y_j^\sharp of Y_j^\flat defined in Remark 4.2. Let n be the unit vector field (with respect to $dq^2 + dw^2$) normal to ℓ_{θ_0} , so n is also normal to ℓ'_{θ_0} and tangent to the lines s^1, s^2 .

Integrating $\partial_n(\partial\phi_j)$ over the rectangle R_{θ_0} and decomposing ℓ_{θ_0} into sets where a given component of $\partial\phi_j$ has constant sign, we obtain that

$$\int_{\ell_{\theta_0}} |\partial\phi_j| \leq \int_{R_{\theta_0}} |\partial^2\phi_j| dqdw + \int_{\ell'_{\theta_0}} |\partial\phi_j|$$

The first integral on the right is bounded above by (7.1). We obtain a uniform upper bound on $\int_{\ell'_{\theta_0}} |\partial\phi_j|$ by proving that $\partial\phi_j$ is bounded above pointwise over ℓ'_{θ_0} , which is true because in Y_j^\sharp we have $\Delta\phi_j = 0$ and $|\phi_j|$ is uniformly bounded. This proves (7.15). \square

Proof of (7.14): Recall that condition vi) implies that $\lim_{j \rightarrow \infty} \sup_{S^i} r|\bar{A}_j| = 0$; using Cauchy-Schwarz, the Hardy inequality and (7.7), we get

$$\begin{aligned} \int_{S^i} |\bar{A}_j|^2 |\partial_1 u_j| d\bar{\mu} &= \int_{S^i} r |\bar{A}_j| \cdot |\bar{A}_j| \cdot \frac{1}{r} |\partial_1 u_j| d\bar{\mu} \\ &\leq \left(\sup_{S^i} r |\bar{A}_j| \right) \left(\int_{S^i} |\bar{A}_j|^2 d\bar{\mu} \right)^{\frac{1}{2}} \left(\int_{S^i} r^{-2} |\partial_1 u_j|^2 d\bar{\mu} \right)^{\frac{1}{2}} \\ &\leq 10 \left(\sup_{S^i} r |\bar{A}_j| \right) \sqrt{M} \rightarrow 0. \end{aligned} \quad (7.16)$$

This proves (7.5), and hence completes the proof of our theorem. \square

8 Examples

In this final section we show that the putative modes of convergence described above actually occur. Namely, we exhibit sequences Y_j of complete, properly embedded minimal surfaces in \mathbb{H}^3 with fixed genus which lose energy in the limit because some portions separate and disappear toward infinity. These Y_j have energy tending to zero and converge smoothly away from a finite number of points on the boundary curve at infinity. The limit is another complete, properly embedded surface Y_* , and we find such sequences where the genus of Y_* is strictly less than that of each of the Y_j . In other words, there can be a loss of genus in the limit. The construction of these surfaces proceeds by a fairly standard gluing result which is directly inspired by one in a recent paper by the second author and Saez [17] (concerning multiple layer solutions of the Allen-Cahn equation in hyperbolic space), and is also quite close to the construction of Maskit combinations' of Poincaré-Einstein metrics by the second author and Pacard [16].

Theorem 8.1. *Choose a finite number, Y_1, \dots, Y_k , of complete, properly embedded minimal surfaces each with finite energy. Suppose that each $\gamma_j = \partial_\infty Y_j$ is a C^2 curve. We assume also that each Y_j is nondegenerate in the sense that it admits no Jacobi fields which decay at γ_j . Then there is a family of complete, properly embedded minimal surfaces Y_t which has boundary curve γ_t a small perturbation of the unit circle. These boundary curves converge in C^2 to the unit circle away from k distinct points q_1, \dots, q_k . Furthermore, there exist rescalings of Y_t at q_j which converge to*

an isometric copy of Y_j . Finally,

$$\mathcal{E}(Y_t) = \sum_{j=1}^k \mathcal{E}(Y_j) + o(1)$$

as $t \rightarrow \infty$.

Proof. The proof is a rather standard gluing theorem, and because of its similarity to the one in [16] we shall sketch it somewhat briefly.

Approximate solutions: The first step is to define a family of surfaces Y'_t which are approximately minimal and have the stated concentration properties. The minimal surfaces Y_t will be small perturbations of these.

First, choose two separate collections of points p_1, \dots, p_k and q_1, \dots, q_k on the unit circle S^1 in the boundary at infinity $\{x = 0\}$, such that no two of these points coincide. For simplicity of notation, assume that $p_j = -q_j$ below. Next, fix points $p'_j, q'_j \in \gamma_j$ and choose a hyperbolic isometry F_j which carries p'_j to p_j and q'_j to q_j , and set $Y'_j = F_j(Y_j)$. Finally, let $M_{j,t}$ be the family of hyperbolic dilations with source p_j and sink q_j , and set $Y_{j,t} = M_{j,t}(Y'_j)$.

As $t \rightarrow +\infty$, the surfaces $Y_{j,t}$ converge locally uniformly in \mathcal{C}^2 in the region $\overline{\mathbb{H}^3} \setminus \{q_j\}$ to the totally geodesic hemisphere H bounded by the unit circle, and this convergence is \mathcal{C}^∞ away from $\{x = 0\}$. In particular, $\gamma_{j,t} := \partial_\infty Y_{j,t}$ converges in \mathcal{C}^2 away from the point q_j . Applying the inverse dilations $M_{j,-t}$, we see that rescalings of Y'_t converge to Y'_j , which is an isometric copy of Y_j .

For each j , choose a closed spherical cap A_j (intersected with the half-space $x \geq 0$) centered at q_j in the unit hemisphere H . We can do this so that these caps are disjoint from one another, and we then let $B_j = H \setminus A_j$. Choose a slightly larger spherical cap $B'_j \supset B_j$, so $B'_j \cap A_j$ is diffeomorphic to a rectangle. Let A'_j be the complement of B'_j in H . By the convergence explained in the last paragraph, some portion $B'_{j,t} \subset Y_{j,t}$ is a normal graph over B'_j with graph function $u_{j,t}$ converging to 0 in $\mathcal{C}^2(B'_j) \cap \mathcal{C}^\infty(B'_j \setminus (B'_j \cap \{x = 0\}))$. Finally, choose a smooth nonnegative cutoff function χ_j which has support in $A_j \setminus (A_j \cap B'_j)$ and which equals 1 in A'_j . Let $Y'_{j,t}$ be the surface which agrees with $Y_{j,t}$ over A'_j and which has graph function $\chi_j u_{j,t}$ over B'_j .

By construction, each $Y'_{j,t}$ coincides with the totally geodesic hemisphere in the region B_j , and this region is disjoint from all of the other regions A_i , $i \neq j$. This means that we may define the surface Y'_t to be the superposition of these k separate surfaces, since they all agree on the complement of the union of the A_j in the hemisphere H .

Observe that these surfaces are minimal in $H \setminus (A_1 \cup \dots \cup A_k)$ and in $A'_1 \cup \dots \cup A'_k$, and the discrepancy from being minimal in the overlap regions tends to 0 as $t \rightarrow \infty$.

Analysis of the Jacobi operator Consider the Jacobi operator

$$L_j = \Delta_{Y_j} + |A_j|^2 - 2$$

on the surface Y_j . This operator has continuous spectrum filling out the half-line $(-\infty, -9/4]$ and a finite number of L^2 eigenvalues above that ray. The assumption that Y_j is nondegenerate means that $L_j : H^2(Y_j) \rightarrow L^2(Y_j)$ is an isomorphism, i.e. 0 is not an L^2 eigenvalue. It is also the case, cf. [1] that under this condition, L_j is an isomorphism on other function spaces better suited for the gluing argument. In particular, let $x^\delta \mathcal{C}^{k,\alpha}$ denote the intrinsic Hölder space (relative to the metric on Y_j induced from the hyperbolic metric) weighted by the function x^δ , where x is the upper half-space coordinate restricted to Y_j . As described carefully in [1], if $0 < \delta < 3$, then

$$L_j : x^\delta \mathcal{C}^{2,\delta}(Y_j) \longrightarrow x^\delta \mathcal{C}^{0,\alpha}(Y_j)$$

is an isomorphism. Denote its inverse by G_j .

Let us now define a family of weighted Hölder spaces on the surfaces Y'_t . We have already defined the cutoff functions χ_j , $j = 1, \dots, k$, and it is clearly possible to add one extra smooth nonnegative function χ_0 which equals 1 on $H \setminus (A_1 \cup \dots \cup A_k)$ and is supported away from $A'_1 \cup \dots, A'_k$, such that $\{\chi_0, \dots, \chi_k\}$ is a partition of unity on Y'_t . (We suppress the dependence on t in the χ_j .) Now define

$$\mathcal{C}_{\delta,t}^{\ell,\alpha}(Y'_t) = \{u = \sum_{j=0}^k \chi_j u_j, \text{ where } u_j = (M_t \circ F_j)^* v_j, \ v_j \in x^\delta \mathcal{C}^{\ell,\delta}(Y_j), \ j = 1, \dots, k, \\ \text{and } u_0 \equiv v_0 \in x^\delta \mathcal{C}^{\ell,\alpha}(H)\}.$$

Notice that $\mathcal{C}_{\delta,t}^{\ell,\alpha}(Y'_t) = x^\delta \mathcal{C}^{\ell,\alpha}(Y'_t)$, but the norm

$$\|u\|_{\delta,t} = \sum_{j=0}^k \|v_j\|_{\ell,\alpha,\delta}$$

in this newly defined space is not uniformly equivalent with respect to the parameter t .

Next, we can transfer the inverse G_j on Y_j using the mapping $M_t \circ F_j$ to an operator $G_{j,t}$ on $Y'_{j,t}$, and then define

$$\tilde{G}_t = \sum_{j=0}^k \tilde{\chi}_j G_{j,t} \chi_j.$$

Here each $\tilde{\chi}_j$ is a nonnegative smooth cutoff function which is equal to 1 on the support of χ_j and vanishes outside a larger neighbourhood. We compute that if L_t denotes the Jacobi operator on Y'_t , then

$$L_t \tilde{G}_t = \text{Id} - \sum_{j=0}^k [L_t \tilde{\chi}_j] G_{j,t} \chi_j := \text{Id} - K_t.$$

The operator K_t is a smoothing operator; this is because the supports of $[L_t, \tilde{\chi}_j]$ and χ_j are disjoint from one another, and because G_j is a pseudodifferential operator, the Schwartz kernel of which is necessarily singular only along the diagonal. Moreover, it is possible to choose the supports of these two functions, $[L_t, \tilde{\chi}_j]$ and χ_j , very far from one another. On the other hand, the Schwartz kernel of $G_{j,t}$ has a decay profile equivalent to the one of G_j ; namely, $G_j(z, z') \leq C \exp(-3 \text{dist}(z, z'))$. Taking these facts together, and arguing exactly as in [19], we conclude that the norm of K_t as a mapping on $\mathcal{C}_{\delta,t}^{\ell,\alpha}$ for any fixed $\delta \in (0, 3)$, can be made as small as desired, uniformly in t , by choosing the supports of these cutoff functions appropriately. We conclude from this that

$$L_t : \mathcal{C}_{\delta,t}^{2,\alpha}(Y'_t) \longrightarrow \mathcal{C}_{\delta,t}^{0,\alpha}$$

is an isomorphism for all $t > 0$ whenever $0 < \delta < 3$, and the norm of its inverse is uniformly bounded in t as $t \rightarrow \infty$.

The gluing construction If ν is the (hyperbolic) unit normal to Y'_t and ϕ is any function on Y'_t , then define the normal graph

$$Y_{t,\phi} = \{\exp_z(\phi(z)\nu(z)) : z \in Y'_t\}.$$

Let \mathcal{M} denote the minimal surface operators on Y'_t , i.e. $\mathcal{M}(\phi)$ is the (hyperbolic) mean curvature function of $Y_{t,\phi}$, viewed as a graph over Y'_t . This is a second order quasilinear operator which can be written as a small perturbation of the minimal surface operators for normal graphs on $Y_{j,t}$ and H , but the main thing we need to know about it is that its linearization at $\phi = 0$ is simply the Jacobi operator L_t .

The perturbation argument is standard. Set $\mathcal{M}(0) = f$. It is not hard to see that $\|f\|_{0,\alpha,\delta} \rightarrow 0$ as $t \rightarrow \infty$. Expand $\mathcal{M}(\phi) = 0$ as

$$f + L_t\phi + Q_t(\phi) = 0 \implies L_t\phi = -f - Q_t(\phi);$$

here Q_t is quadratic remainder term involving the terms ϕ , $\nabla\phi$ and $\nabla^2\phi$ which satisfies

$$\|Q_t(\phi)\|_{0,\alpha,\delta} \leq C\|\phi\|_{2,\alpha,\delta}^2$$

and

$$\|Q_t(\phi) - Q_t(\psi)\|_{0,\alpha,\delta} \leq C(\|\phi\|_{2,\alpha,\delta} + \|\psi\|_{2,\alpha,\delta})\|\phi - \psi\|_{2,\alpha,\delta}.$$

The equation

$$\phi = -G_t(f + Q_t(\phi)),$$

can then be solved using the estimates above by a straightforward contraction mapping argument.

It is easy from the construction to see that if t is quite large, then $\|\phi\|_{2,\alpha,\delta}$ is small and the surface $Y_t := Y_{t,\phi}$ is embedded. Since $\phi \rightarrow 0$ at $\partial_\infty Y'_t$, we see that the new surface has the same boundary curve at infinity. The fact that Y_t converges in C^2 away from the points q_1, \dots, q_k follows directly from the construction. \square

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